Iterations of the Lévy C Curve

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January 2018, Draft 4

Abstract

Various properties of finite iterations of the Lévy C curve, including coordinates, boundary, area, squares, centroid, moment of inertia, and weight in regions.

Contents

Notation

A few formulas have terms going in a repeating pattern of say 4 values according as an index $k \equiv 0 \text{ to } 3 \bmod 4$. They are written like

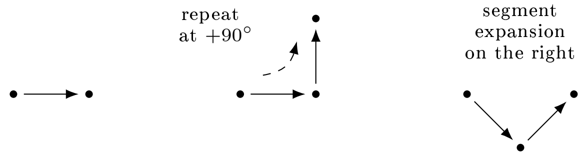
$$[5,\ 8,\ -5,\ 9] \qquad \text{values according as } k \bmod 4$$

meaning 5 when $k \equiv 0 \bmod 4$, or 8 when $k \equiv 1 \bmod 4$, etc. Likewise periodic patterns of other lengths, usually at most 8.

Periodic patterns like this can be expressed by powers of -1 or i (or other roots of unity), but except in simple cases that tends to be less clear than the values.

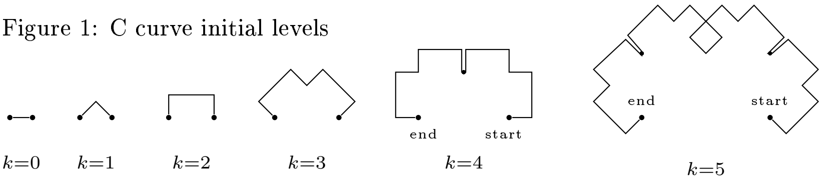
1 C Curve

The C curve by Lévy[3] is defined as repeated copying of itself at 90° angles beginning from a unit line segment. Or equivalently an expansion of each line segment to a pair of segments.



Each segment expands on the right side. The usual expansion angle is 90° and with that the curve variously touches, overlaps and crosses itself. In the following diagrams the segment overlaps in k=4 and k=5 are shown with a little separation. The top of k=5 is a crossing.

Figure 1: C curve initial levels



The curve is drawn above with the start and end horizontal. It can also be drawn with the first segment in a fixed direction. It that case it spirals around anti-clockwise (the usual mathematical direction).

k=6 with the start a fixed direction East shows the "C" shape. The middle overlap is a combination overlap and crossing. The 2×2 square is traversed anti-clockwise.

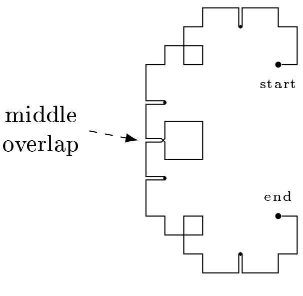
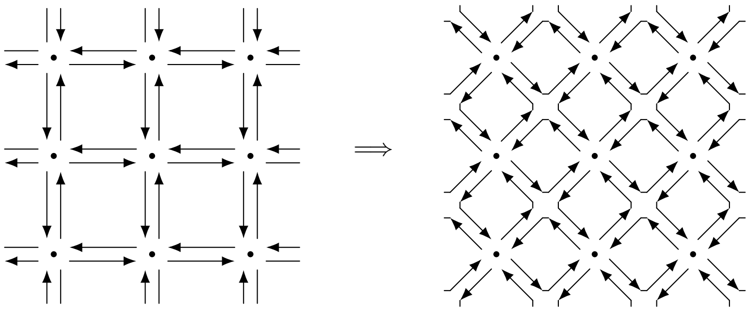


Figure 2:
k=6

Theorem 1. *The C curve traverses a given segment at most once forward and once backward.*

Proof. Consider an infinite square grid with line segments connecting the points, once forward and once backward. Each line segment expands on the right as

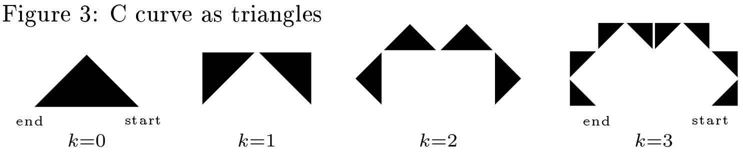


The expanded segments are the same grid pattern rotated 45°.

Any subset of the full grid with at most one forward and one backward expands to a new bigger set with the same property. The C curve begins as a single line segment which is such a subset. ☐

Line segments can also be considered as triangles extending on the right side to fill a quarter of the adjacent square. The grid of lines above is then a tiling of the plane by triangles and their expansion is a new tiling.

Figure 3: C curve as triangles



Each triangle expands by dividing into two halves and flipping outwards across the sides. The total area is the same at all expansion levels but spreads out.

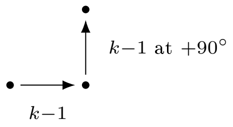
The short sides of each triangle are where the line segment expands to in the next level. Some of the triangle measures like counts of sides become functions of segments in level k and k+1.

1.1 Direction

Theorem 2. *Number segments of the C curve starting n=0 for the first and take its direction to be d=0. The direction of segment number n is the number of 1-bits of n written in binary.*

$$\begin{aligned} \mathit{dir}(n) &= \mathit{CountOneBits}(n) \\ &= 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4, 1, 2, 2, 3, \ldots \qquad \text{A000120} \end{aligned}$$

Proof. As described above a level k curve is comprises level $k-1$ and a copy of level $k-1$ turned +90°,



$$\mathit{dir}(n + 2^k) \;=\; \mathit{dir}(n) + 1 \qquad \text{starting } \mathit{dir}(0) = 0$$ ☐

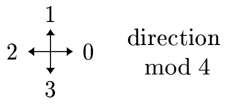
For example N=11 is binary “1011” which is three 1-bits, so direction $3 \times 90^\circ = 270^\circ$, ie. to the south. $n{=}2^k$ has only a single 1-bit so $dir(2^k) = 1$. This is the first segment of the next replication and is simply the first segment East rotated $+90^\circ$ to be North.

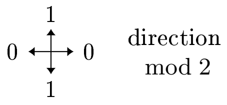
Direction can be taken mod4 for a net direction

$$dir(n) \bmod 4 = 0,1,1,2,1,2,2,3,1,2,2,3,2,3,3,0,1,2,2,3,\ldots \qquad \text{A179868}$$

Or taken mod 2 is the Thue-Morse parity sequence. A horizontal segment is $dir(n) = 0 \bmod 2$. A vertical segment is $dir(n) = 1 \bmod 2$.

$$\begin{aligned} dir(n) \bmod 2 &= 0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0,1,0,0,1,\ldots \qquad \text{A010060} \\ &= \begin{cases} 0 & \text{if segment horizontal} \\ 1 & \text{if segment vertical} \end{cases} \end{aligned}$$





1.2 Turn

At each vertex the curve can turn in any direction: left, right, straight, or 180° back.

Theorem 3. *Number the vertices of the C curve starting from* $n{=}0$ *for the start so the first turn is at* $n{=}1$*. The turn is given by the number of low 0-bits when* n *is written in binary,*

$$\begin{aligned} turn(n) &= dir(n) - dir(n{-}1) \qquad n \ge 1 \\ &= 1 - CountLowZeros(n) \\ &= 1,0,1,-1,1,0,1,-2,1,0,1,-1,1,0,1,-3,\ldots \qquad n{\ge}1 \ \text{A088705} \end{aligned}$$

$$CountLowZeros(n) = 0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4,\ldots \qquad \text{A007814}$$

Proof. The directions of n and $n{-}1$ are their 1-bit count. The two differ by some low bits

$$\begin{array}{rl} n{-}1 & \boxed{\cdots \mid 0 \mid 1\cdots1} \\[1ex] n & \boxed{\cdots \mid 1 \mid 0\cdots0} \end{array}$$

From $n{-}1$ to n the number of 1 bits decreases by $CountLowZeros(n)$ for the 1s which become 0s, and increases by one for the $0{\to}1$ above them. ☐

For example $n = 8$ is binary “100” which is 2 low 0-bits for $turn(8) = 2-1 = 90^\circ$, to the right.

When n is odd there are no low zero bits and $turn = 1$ so every second turn is left by 90°. The other turns are straight or right by some amount (including possibly all the way around to be left again).

The turn at the next point can be calculated in a similar way by low 1-bits. This has the effect of numbering turns starting from 0. If segments are numbered starting from 0 then this is the turn after the segment.

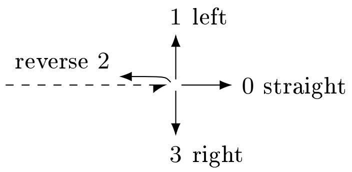
$$\begin{aligned} \mathit{turn}(n{+}1) &= \mathit{dir}(n{+}1) - \mathit{dir}(n) \qquad\qquad n \ge 0 \\ &= 1 - \mathit{CountLowOnes}(n) \end{aligned}$$

n	$\cdots$	0	$1\cdots1$
$n{+}1$	$\cdots$	1	$0\cdots0$

Consecutive values in the turn sequence are always different since $\mathit{turn}(n) = 1$ for n odd but $\mathit{turn}(n) < 1$ for n even.

The turn can be taken mod 4,

$$\mathit{turn4}(n) = 1,0,1,3,1,0,1,2,1,0,1,3,1,0,1,1,1,0,1,3,1,0,\ldots$$



$\mathit{turn} \equiv 0,1,2,3$
mod 4

In this form there are 3 consecutive left turns at $n{-}1, n, n{+}1$ whenever $\mathit{CountLowOnes}(n) \equiv 0 \bmod 4$. The first of these is at $n = 15, 16, 17$ which is the diamond in $k{=}5$ of figure 1.

A "morphism" generating *turn4* is to simultaneously substitute in the sequence

$$0 \to 1{,}3 \qquad 1 \to 1{,}0 \qquad 2 \to 1{,}1 \qquad 3 \to 1{,}2$$

The first of each pair is odd n (due to starting at $n{=}1$) and the second even. The substitutions are as simple as a left turn 1 for every odd n and the existing turn decreasing for even n for the new low 0-bit on $2n$,

$$\mathit{turn}(2n) = \mathit{turn}(n) - 1 \qquad\qquad \mathit{turn}(2n{+}1) = 1 \tag{1}$$

The geometric interpretation is that the segment expansions at a given turn location are an extra 90° to the right, which is -1.

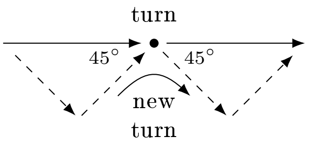


Figure 4:
turn change
on segment expansion

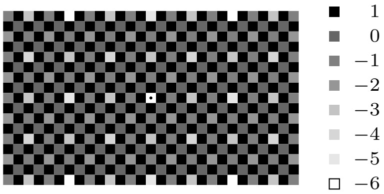
Applying the morphism 4 times brings it back to the existing turn unchanged. So the *turn4* sequence is itself with a fixed 15 turn sequence inserted at the start, end, and between each existing turn.

101310121013101		101310121013101		101310121013101		101310121013101	new turns
	1		0		1		existing turns

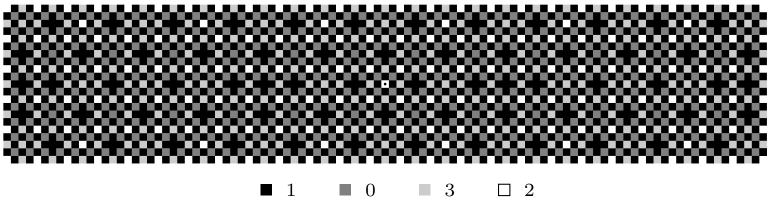
The expansion rule (1) also gives the locations of turns. An odd n is an odd location $z \equiv 1 \bmod b$, meaning $z = x{+}iy$ with $x{+}y$ odd. Subsequent expansions are multiply by b so that after k expansions $z \equiv b^k \bmod b^{k+1}$ and this k =

CountLowZeros(n) for *turn*. When the curve variously overlaps all the n at a given point have the same turn because all reached there by expanding at the same time from odd n.

1	-1	1	-4	1	-1	1	-3	1	-1	1	-4	1	-1	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	-2	1	-1	1	-2	1	-1	1	-2	1	-1	1	-2	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	-1	1	-3	1	-1	1	•	1	-1	1	-3	1	-1	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	-2	1	-1	1	-2	1	-1	1	-2	1	-1	1	-2	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	-1	1	-4	1	-1	1	-3	1	-1	1	-4	1	-1	1

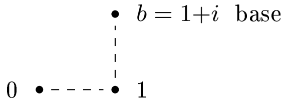


Or similarly turns mod 4 at location,



1.3 Coordinates

It's convenient to calculate C curve coordinates in complex numbers using a base b which is the end of a segment expansion.



Number points along the curve starting from 0 at the origin. The location of point n can be found by writing it in binary and using those bits in a sum of b powers and directions.

$$
\begin{aligned}
n &= 2^k a_k + 2^{k-1} a_{k-1} + \cdots + 2a_1 + a_0 \qquad\qquad a_j = 0 \,\text{or}\, 1 \\
&= \text{binary } a_k\, a_{k-1} \ldots a_1\, a_0
\end{aligned}
$$

$$
\begin{aligned}
\mathit{point}(n) = b^k \quad & a_k && \text{high bit} \\
+\, b^{k-1} \; & a_{k-1} \; i^{\,\mathit{dir}(a_k)} \\
+\, b^{k-2} \; & a_{k-2} \; i^{\,\mathit{dir}(a_k\, a_{k-1})} \\
\ldots \\
+\, b^1 \quad & a_1 \quad i^{\,\mathit{dir}(a_k\, a_{k-1} \ldots a_2)} \\
+\, b^0 \quad & a_0 \quad i^{\,\mathit{dir}(a_k\, a_{k-1} \ldots a_2\, a_1)} && \text{low bit} \\
= 0,\ 1,\ 1{+}i,\ 1{+}2i,\ 2i,\ & 3i,\ -1{+}3i,\ -2{+}3i,\ -2{+}2i, \ldots
\end{aligned}
\tag{2}
$$

Each *dir* uses the bits above. This is the direction function from subsection 1.1 applied to those bits. For any zero bit $a_j{=}0$ there is no change to the direction so the effect is an extra i power at each 1-bit.

$$\begin{aligned} n &= 2^{k_0} + 2^{k_1} + \cdots + 2^{k_t} \qquad k_0 > k_1 > \cdots > k_t \\ point(n) &= i^0 b^{k_0} + i^1 b^{k_1} + \cdots + i^t b^{k_t} \end{aligned} \tag{3}$$

Bits can be taken high to low, with n_{k-1} as the bits below the high a_k. Here i^{a_k} is the extra i power.

$$point(2^k a_k + n_{k-1}) \;=\; b^k\, a_k \;+\; point(n_{k-1})\,.\, i^{\,a_k}$$

Bits can be taken low to high,

$$point(2n_1 + a_0) \;=\; point(n_1).b \;+\; i^{\,dir(n_1)}.a_0$$

a_0 is the low bit and n_1 the bits above it. $dir(n_1)$ depends on all the bits of n_1 but there's no need to calculate that in full. It's enough to form a direction factor $dir(a_1)$, $dir(a_2)$ etc for each successive bit and apply to all lower terms, as if evaluating outwards a nested expression like

$$\begin{aligned} point(n) = {} & b^k\, a_k \\ & + i^{\,a_k}\big(b^{k-1}\, a_{k-1} \\ & \qquad \cdots \\ & \qquad + i^{\,a_2}\big(b^1\, a_1 \\ & \qquad\qquad + i^{\,a_1}\big(b^0\, a_0\,\big)\big)\big) \end{aligned}$$

point(n) can be reversed low to high to calculate n for a given segment. Suppose a segment is at $z = point(n)$ and direction $d = dir(n)$ mod 4.

```
unpoint($z$,  $d = 0,1,2,3$)
  **loop** until $z = 0$ or $\pm 1$ or $\pm i$
    $bit = z$ mod $b$          0 or 1, bits of $n$ low to high
    **if** $bit = 1$  **then**
      $d \leftarrow d-1$          rotate $-90$ deg
      $z \leftarrow z - i^d$          move to multiple of $b$
    **end if**
    $z \leftarrow z/b$        $z$ is a multiple of $b$, divide out
  **end loop**

  **if** $d$=0 and $z$=0 **then** found $n$
  **if** $d$=1 and $z$=1 **then** found $n$, with extra high 1-bit
  **otherwise** no such segment $z, d$ in curve
```

A given $z = x + iy$ is a multiple of b when $x{+}y$ is even, since any $(u + iv)b = (u{-}v) + (u{+}v)i$ has sum $\mathrm{Re} + \mathrm{Im} = 2u$. So the low bit a_0 is determined by $z \equiv 0 \bmod b$ or $z \equiv 1 \bmod b$ which is $x{+}y \equiv 0 \,\mathrm{or}\, 1 \bmod 2$.

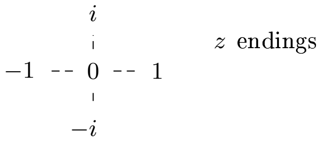
The subtraction $z - i^d$ removes the low term of the *point* formula (2). The direction original $d = dir(a_k a_{k-1} \ldots a_1 a_0)$ is changed to $d = dir(a_k a_{k-1} \ldots a_1)$ by subtracting 1 when low a_0=1. Then dividing z/b leaves z, d as the bits above and the procedure can be repeated.

The loop reduces z by dividing b each time except for the $-i^d$ offset. Considering just magnitudes, $|z|$ decreases when

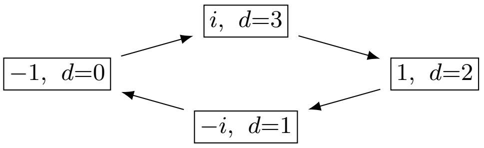
$$|z| - \left|\frac{z - i^d}{b}\right| \;\ge\; |z| - \frac{|z| + 1}{\sqrt{2}} \;=\; \left(1 - \tfrac{1}{2}\sqrt{2}\right)|z| - \tfrac{1}{2}\sqrt{2}$$

$$> 0 \quad \text{when} \quad |z| > 1+\sqrt{2}$$

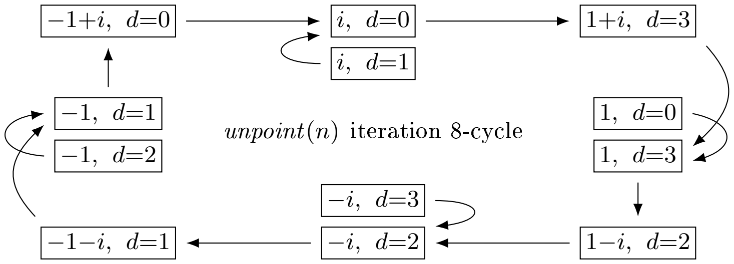
For $|z| \le 1+\sqrt{2}$ it can be verified explicitly that all integer z reaches one of the loop ends $0, \pm 1, \pm i$ for any d. These loop ends are one of the five points



It's not possible to wait for $z{=}0$ because some d directions will step infinitely in a cycle among the 4 non-zeros. There is a cycle of 4, and a cycle of 8 going $\pm 1 \pm i$. All of these are no such segment, in that no finite set of a bits will give the original z. (The odd z in these cycles give $bit{=}1$ successively.)



unpoint(n) iteration 4-cycle



unpoint(n) iteration 8-cycle

The case $z{=}1, d{=}1$ goes to $z{=}0$ on one further iteration. This is $bit{=}1$ and hence its extra high bit. That bit could be handled by letting the loop continue when $z{=}1, d{=}1$. But $z{=}1, d = 0, 2, 3$ must terminate since they are in the cycles shown above.

$z{=}i^{d-1}, d{=}0, 2, 3$ reach $z{=}0$ on one further iteration too, but with $d{\ne}0$ so no such segment.

The curve visits a given z location up to 4 times and the four $d = 0, 1, 2, 3$ give those n. d is the direction the segment leaves the point. So to find all n at a given z each d direction should be attempted. Locations which are not visited, or which are visited but a segment does not leave in direction d result in "no such segment".

Attempting $d = 0, 1, 2, 3$ gives n with $\mathit{CountOneBits} \equiv d \bmod 4$ but the magnitudes of these n can be in any order. There are points in the curve with each possible ordering and combination of no such d segment. First occurrences range up to $n{=}95583$ (which is a 4-visit in $k{=}18$).

For computer calculation everything can be done in Cartesian coordinates $x{+}iy = z$, without full complex number arithmetic. Each bit is $x{+}y \bmod 2$ and the division is $(x, y) \leftarrow (\frac{x+y}{2}, \frac{y-x}{2})$ in the usual way.

An alternative is to calculate in terms of sum $s = x{+}y$ and difference $m = y{-}x$. The loop end condition becomes $-1 \le s \le 1$ and $-1 \le m \le 1$ which might be more convenient than 5 specific x, y combinations.

$$\begin{array}{ccc} -1,1 & \cdot & 1,1 \\ \cdot & 0,0 & \cdot \\ -1,-1 & \cdot & 1,-1 \end{array}$$

s, m in ranges
$-1 \leq s \leq 1$
$-1 \leq m \leq 1$

Each bit for s, m is then $bit \equiv s \bmod 2$ and the division is the same $(s,m) \leftarrow (\frac{s+m}{2}, \frac{m-s}{2})$. The effect is to work with bz, ie. an extra factor of b throughout.

1.4 Segments in Direction

Theorem 4. *Take the first segment of the C curve as direction $d{=}0$. The number of segments in each direction $d = 0,1,2,3$ for level k is*

$$\begin{aligned}
S(k,d) &= 1,0,0,0 \quad \textit{for } d \equiv 0 \,\text{to}\, 3 && \textit{for } k=0 && (4)\\
&= \tfrac{1}{4}\Big(2^k + s(k{-}2d)\,.\,2^{\lfloor k/2 \rfloor}\Big) && \textit{for } k \geq 1 && (5)\\
&= \tfrac{1}{4}\Big(2^k + b^k(-i)^d + \overline{b^k(-i)^d}\Big) && \textit{for } k \geq 1 && (6)\\
&= \tfrac{1}{4}\Big(\,\big|b^k + i^d\big|^2 + 1\Big) && \textit{for } k \geq 1 &&\\
s(m) &= [2,\,2,\,0,\,-2,\,-2,\,-2,\,0,\,2] && && 2{\times}\text{A046980}\\
S(k,0) &= 1,1,1,1,2,\ \ 6,16,36,72,136,256,\ldots && && \text{A038503}\\
S(k,1) &= 0,1,2,3,4,\ \ 6,12,28,64,136,272,\ldots && && \text{A038504}\\
S(k,2) &= 0,0,1,3,6,10,16,28,56,120,256,\ldots && && \text{A038505}\\
S(k,3) &= 0,0,0,1,4,10,20,36,64,120,240,\ldots && && \text{A000749}
\end{aligned}$$

Proof. The segments in direction $d{=}0$ are those n which have $\operatorname{dir}(n) \equiv 0 \bmod 4$. So a count 0, 4, 8, 12, etc many 1-bits among total k bits. Similarly other d. The possible positions for $d \bmod 4$ many 1-bits are a binomial coefficient.

$$\begin{aligned}
S(k,0) &= \tbinom{k}{0} + \tbinom{k}{4} + \tbinom{k}{8} + \tbinom{k}{12} + \cdots\\
S(k,1) &= \tbinom{k}{1} + \tbinom{k}{5} + \tbinom{k}{9} + \tbinom{k}{13} + \cdots\\
S(k,2) &= \tbinom{k}{2} + \tbinom{k}{6} + \tbinom{k}{10} + \tbinom{k}{14} + \cdots\\
S(k,3) &= \tbinom{k}{3} + \tbinom{k}{7} + \tbinom{k}{11} + \tbinom{k}{15} + \cdots\\
S(k,d) &= \sum_{j=d,d+4,\ldots} \tbinom{k}{j}
\end{aligned}$$

These sums are from Cournot[1] (and Ramus[4]). Form (5) is a half power to emphasise the result is always an integer. □

These counts are the same in the Heighway/Harter dragon curve, but the order of the segment directions and the resulting shape are not the same for $k \geq 2$.

Theorem 5. *Among the first n segments of the C curve, the number in direction $d \bmod 4$ is*

$$SN(n,d) = \tfrac{1}{4}\Big(n + 2\operatorname{Re}(-i)^d \mathit{point}(n) \;-\; \big((-1)^{d+\mathit{dir}(n)} \;\; \textit{if } n \textit{ odd}\big)\Big) \qquad (7)$$

$$
\begin{aligned}
SN(n,0) &= 0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,\dots\\
SN(n,1) &= 0,0,1,2,2,3,3,3,3,4,4,4,4,4,4,4,4,\dots\\
SN(n,2) &= 0,0,0,0,1,1,2,3,3,3,4,5,5,6,6,6,6,\dots\\
SN(n,3) &= 0,0,0,0,0,0,0,0,1,1,1,1,2,2,3,4,4,\dots
\end{aligned}
$$

Proof. Each pair of segments $2j, 2j{+}1$ is one horizontal and one vertical. This follows from segment expansion making pairs horizontal and vertical, or from one extra bit in $2j{+}1$ making *dir* opposite parity. When n even the total segments in directions 0 and 2 is therefore $n/2$. When n odd if the direction $dir(n)$ is odd then the segment after the first n is vertical so the one before is horizontal.

$$SN(n,0) + SN(n,2) = n/2 - \left(\tfrac{1}{2}(-1)^{dir(n)} \text{ if } n \text{ odd}\right) \qquad \text{total horizontal} \tag{8}$$

The difference between numbers of segments 0 and 2 is horizontal position,

$$SN(n,0) - SN(n,2) = \operatorname{Re} point(j) \qquad \text{net horizontal} \tag{9}$$

(8)+(9) and (8)−(9) give directions 0 and 2 separately. Similarly for the verticals, with Im and the $dir(n)$ part + instead. In (7) the $(-i)^d$ selects $\pm\operatorname{Re},\operatorname{Im}$ and $(-1)^d$ selects ± of the *dir* part. ☐

Second Proof of Theorem 5. SN can be written in terms of the whole level counts S. Suppose the bits in n are

$$n = 2^{k_0} + 2^{k_2} + \cdots + 2^{k_t} \qquad k_0 > k_1 > \cdots > k_t$$

The first 2^{k_0} segments of n are $S(k_0, d)$. Then the rest of n is rotated +90° so desired direction $d{-}1$ for counts hence $S(k_1, d{-}1)$. An so on,

$$SN(n,d) = S(k_0,d) + S(k_1,d{-}1) + \cdots + S(k_t,d{-}t)$$

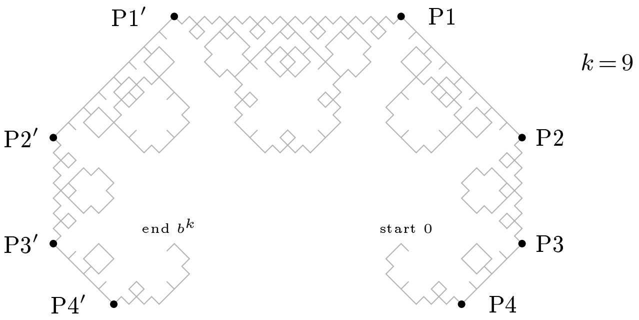
For $d{=}0$, the 2^k part of S at (6) for each term sums to n. The b^k parts and successive $d{-}1$ down to $d{-}t$ is the same as *point* at (3), for $k_t \geq 1$. Or rather the same but with conjugates which make no difference since both plain and conjugate are taken in (6) giving the real part only.

If $k_t = 0$, which is n odd, the S case $k{=}0$ at (4) is not this 2^k and b^k pattern but instead a fixed 1 in direction 0. Whether direction $d{-}t = 0$ is given by $dir(n)$ which is the count of bits k_0 to k_t.

For $d{\neq}0$ similarly with the 2^k parts the same and a factor $(-i)^d$ to rotate *point* to the Re direction, and suitable final low bit case when n odd. ☐

2 Convex Hull

A convex hull is the smallest convex polygon which can be drawn around a given set of points.



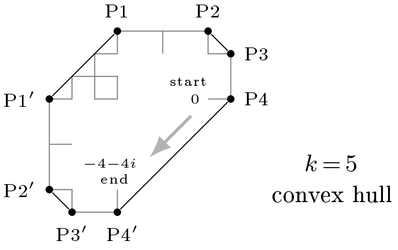
Theorem 6. *The convex hull around C curve $k \geq 5$ is a set of 8 vertices*

$$\begin{aligned} P1_k &= -i\,b^k - i^{\lfloor k/2 \rfloor} & P1'_k &= b^k - i.\overline{P1_k} \\ P2_k &= P1_{k-1} & P2'_k &= b^k - i.\overline{P2_k} \\ P3_k &= P1_{k-2} & P3'_k &= b^k - i.\overline{P3_k} \\ P4_k &= P1_{k-3} & P4'_k &= b^k - i.\overline{P4_k} \end{aligned}$$

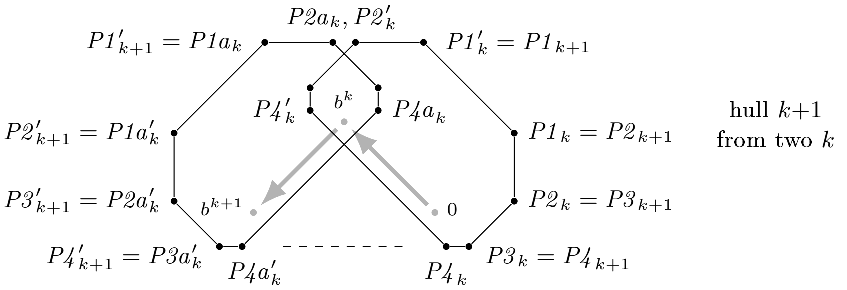
For $k < 5$ these points are the hull vertices but with some duplications and some points excluded.

k	*vertices*	*duplication*	*exclude*
0	*2*	$(P3 = P3' = \frac{1}{2}$ *on boundary)*	$P2, P4, P2', P4'$
1	*3*	P1=P1′ (P4=$\frac{1}{2}$ *on boundary)*	$P3, P3', P4'$
2	*4*	P1=P2 *and* P1′=P2′	$P4, P4'$
3	*6*	P2=P3 *and* P3=P4	
4	*6*	P3=P4 *and* P3′=P4′	

Proof. Hulls for $k \leq 5$ can be constructed explicitly and are per the formulas. The hull for k=5 has successive 45° sides,



Curve k+1 is two k at right angles to the hull around k+1 is formed the hulls around two k at right angles. In the following diagram the second k is “a”.



The two k hull sides at the top are co-linear by symmetry. From the formulas the $P1'_k$ point is right of $P2a_k$, and similarly $P1a_k$, so that $P1'_k$ and $P1a_k$ are new hull vertices.

The bottom two k sides are also co-linear by symmetry, and again $P3_k$ is right of $P4a'_k$ so that $P3_k$ and $P3a'_k$ are new hull vertices. ☐

The two $P2a_k, P2'_k$ vertices at the top are equal when k+1 even or a unit square diagonal apart when k+1 odd.

Hull vertex limits for the curve scaled to a unit length are the coefficients of b^k in P1 etc. The extents are 1 top, $\frac{1}{4}$ bottom, and $\frac{1}{2}$ each end.

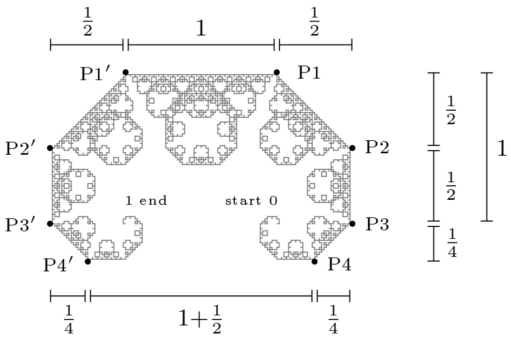


Figure 5: hull limit extents

The hull boundary length is sum of sides $|P1_k - P2_k|$ etc.

$$HB_k = \begin{cases} 2,\ \ 2{+}\sqrt{2},\ \ 6 & \text{if } k = 0 \text{ to } 2 \\ (\frac{7}{2}{+}\frac{3}{2}\sqrt{2}).\sqrt{2}^k \ - 4\sqrt{2} & \text{if } k \geq 3 \end{cases}$$
$$= 2,\ \ 2{+}\sqrt{2},\ \ 6,\ \ 6{+}3\sqrt{2},\ \ 14{+}2\sqrt{2},\ \ 12{+}10\sqrt{2},\ \ldots$$
$$\frac{HB_k}{\sqrt{2}^k} \to \frac{7}{2} + \frac{3}{2}\sqrt{2} \ \ = 5.621320\ldots \tag{10}$$

The rational and $\sqrt{2}$ parts of HB separately are

$$HB_k = HBrat_k + HBsqrt_k\sqrt{2}$$
$$HBrat_k = \begin{cases} 2, 2, 6 & \text{if } k = 0 \text{ to } 2 \\ \frac{1}{2}\,[7, 6]\,2^{\lfloor k/2 \rfloor} & \text{if } k \geq 3 \end{cases}$$
$$= 2, 2, 6, 6, 14, 12, 28, 24, 56, 48, 112, \ldots$$

$$HBsqrt_k = \begin{cases} 0,1,0,3 & \text{if } k = 0 \,\text{to}\, 3 \\ HBrat_{k-1} - 4 & \text{if } k \geq 4 \end{cases}$$
$$= 0,1,0,3,\ \ 2,10,\ \ 8,24,20,52,\ \ 44,\ldots$$

The area of the hull can be calculated from its vertices as triangular areas. HA_1 and HA_3 are fractions and all others are all integers.

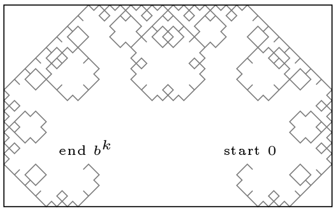
$$HA_k = \begin{cases} 0,\, \frac{1}{2},\, 2 & \text{if } k = 0 \,\text{to}\, 5 \\ \frac{35}{16} 2^k - \frac{1}{2}[10,13].2^{\lfloor k/2 \rfloor} + 2 & \text{if } k \geq 3 \end{cases}$$
$$= 0, \tfrac{1}{2}, 2, \tfrac{13}{2}, 17, 46, 102, 230, 482, 1018, \ldots$$

Theorem 7. *The minimum area rectangle around C curve k has area*

$$MR_k = \begin{cases} 0,1,2 & \textit{if } k = 0 \,\text{to}\, 2 \\ \frac{5}{2} 2^k - \frac{13}{2} 2^{\left\lfloor \frac{k}{2} \right\rfloor} + [4,2] & \textit{if } k \geq 3 \end{cases}$$
$$= 0, 1, 2, 9, 18, 56, 112, 270, 540, 1178, \ldots$$

This minimum rectangle is aligned to the curve endpoints, except for $k{=}1$ *and* $k{=}3$ *where a square at* 45° *is equal minimum area.*

Proof. Any minimum area rectangle has at least one side aligned to a side of the convex hull. So for the C curve there are two alignments. Firstly rectangle aligned to the endpoints,

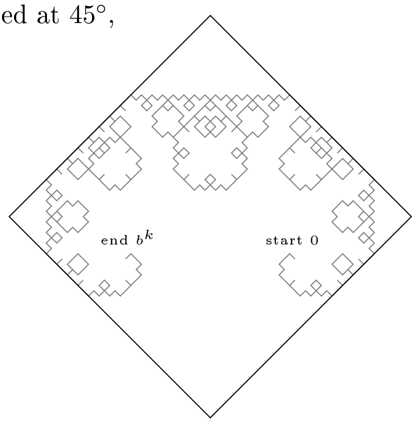


MR_k rectangle aligned to curve endpoints

The sides lengths from the hull vertices is

$$MR_k = \operatorname{Re} \omega_8^{-k} \big(\mathit{P2}_k - \mathit{P2}'_k\big)\ .\operatorname{Im} \omega_8^{-k} \big(\mathit{P4}_k - \mathit{P1}_k\big) \qquad \text{for } k{\geq}3$$
$$\text{where } \omega_8 = e^{2\pi i/8} \ = \tfrac{1}{\sqrt{2}}(1{+}i) \text{ eighth root of unity at } {+}45°$$

Secondly aligned at 45°,



MS_k square at 45°
to curve endpoints

This alignment is a square by symmetry since the top two sides are the same. Likewise the bottom two are the same. The side lengths are again from the hull vertices, and including cases $k < 3$,

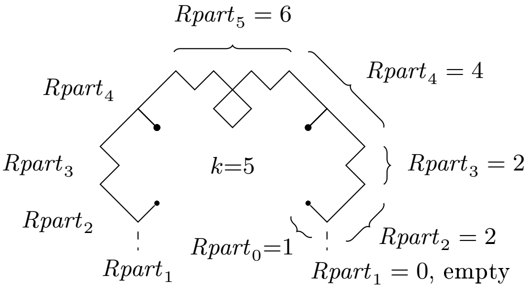
$$\begin{aligned} MS_k &= \Big(\operatorname{Re} \omega_8^{-(k+1)} \big(P4'_k - P1_k\big)\Big)^2 && k \geq 3 \\ &= \Big(\tfrac{5}{4}\sqrt{2}.\sqrt{2}^k - [\sqrt{2}, 2]\Big)^2 && k \geq 3 \\ &= \begin{cases} \frac{1}{2}, 1 & \text{if } k = 0, 1 \\ \frac{25}{8} 2^k - 5.2^{\lceil k/2 \rceil} + [2, 4] & \text{if } k \geq 2 \end{cases} \\ &= \tfrac{1}{2}, 1, \tfrac{9}{2}, 9, 32, 64, 162, 324, 722, 1444, \dots \end{aligned}$$

$MS_1 = 1$ and $MS_3 = 9$ are equal to MR and MS is bigger for $k \geq 4$. The difference is, adapting ceil half power in MS to floor the same as MR,

$$\begin{aligned} MS_k - MR_k &= \tfrac{5}{8} 2^k + [\tfrac{3}{2}, -\tfrac{7}{2}].2^{\lfloor k/2 \rfloor} + [-2, 2] && k \geq 3 \\ &> 0 \qquad \text{for } k \geq 4 \end{aligned}$$

□

3 Right Boundary



k=5
part
lengths

Theorem 8. *The right-side boundary length of C curve level k is*

$$\begin{aligned} R_k &= [7, 10].2^{\left\lfloor \frac{k}{2} \right\rfloor} - 2k - 6 \\ &= 1, 2, 4, 8, 14, 24, 38, 60, 90, 136, 198, 292, \dots \end{aligned}$$

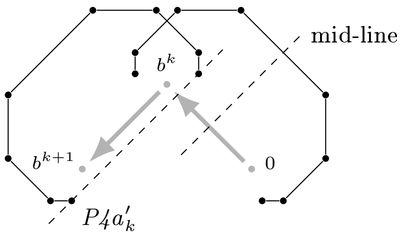
odd $n \geq 1$ A131064

Proof. Take a "part" of the curve as right boundary either straight or zig-zag. Take the middle top of curve k as Rpart_k. This top part is the convex hull top side. It is straight segments when k even or zig-zag when k odd.

These sides are continuous since as noted after theorem 6 the top $P2a_k, P2'_k$ vertices are the same or diagonal step apart. So number of segments is a "Manhattan" distance

$$\begin{aligned}
\mathit{Manhattan}(z) &= |\operatorname{Re} z| + |\operatorname{Im} z| \\
\mathit{Rpart}_k &= \mathit{Manhattan}\big(P1_k - P1'_k\big) \\
&= 2^{\lceil k/2 \rceil} - [0,2] \\
&= 1, 0, 2, 2, 4, 6, 8, 14, 16, 30, 32, 62, \ldots && k{\geq}3 \;\; 2{\times}\mathrm{A097110}
\end{aligned}$$

The part sides curl inwards but do not overlap. This is so of $k{=}5$ and supposing it true of k then the $k{+}1$ curve from two k is

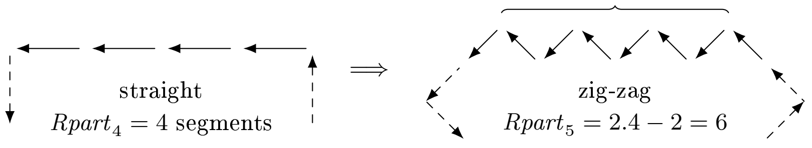


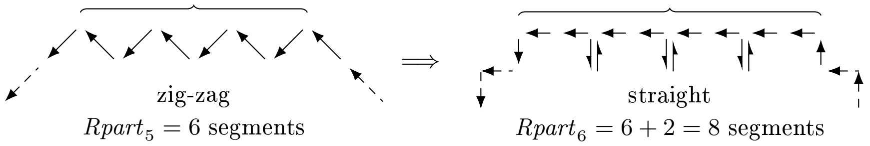
Within k the curling does not overlap so k right boundary does not touch its curve mid-line. Working through the formulas, the second curve bottom hull side $P4a_k$ to $P4a'_k$ is above this mid-line of the first. So curling at the start of the first k is not overlapped in $k{+}1$. Likewise by symmetry at the end of the second k which is the $k{+}1$ end.

The whole boundary is then sum of parts, being a single middle k then two of $k{-}1$ etc on each side.

$$R_k = \mathit{Rpart}_k + 2\sum_{j=0}^{k-1} \mathit{Rpart}_j \qquad \square$$

A straight or zig-zag right boundary part expands respectively as





In a straight part each segment becomes a zig-zag notch, except 1 segment at each end. In a zig-zag part each segment of a notch becomes a straight segment and an extra 1 at each end. This gives a recurrence

$$Rpart_k = \begin{cases} Rpart_{k-1}+2 & \text{if } k \text{ even, zig-zag becomes straight} \\ 2\,Rpart_{k-1}-2 & \text{if } k \text{ odd, straight becomes zig-zag} \end{cases} \tag{11}$$
$$\text{starting } Rpart_0 = 1$$

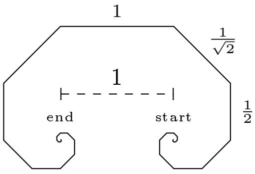
For k=0, $Rpart_0 = 1$ is the single curve segment. For k=1 the two segments are $Rpart_0$ sides and in between the top middle is an empty $Rpart_1 = 0$.

The boundary length doubles for R_0 through R_3 but at $R_4 = 14$ there are two segments not on the right boundary.

R is segments so the diagonals are bigger than their geometric length. For that reason if scaled by $1/\sqrt{2}^k$ for endpoints a unit length apart then the even and odd cases of R do not converge to the same value (but to 7 and $5\sqrt{2}$).

The geometric lengths are

$$\begin{aligned} RgeomPart_k &= \left|P1_k - P1'_k\right| \quad = \sqrt{2}^k - [0, \sqrt{2}] \\ &= \begin{cases} Rpart_k & \text{if } k \text{ even} \\ \frac{1}{2}\sqrt{2}\,Rpart_k & \text{if } k \text{ odd} \end{cases} \\ Rgeom_k &= RgeomPart_k + 2\sum_{j=0}^{k-1} RgeomPart_j \\ &= \left(3{+}2\sqrt{2}\right)\sqrt{2}^k - k\sqrt{2} - 2 - 2\sqrt{2} \\ &= 1,\ 2,\ 4,\ 6{+}\sqrt{2},\ 10{+}2\sqrt{2},\ 14{+}5\sqrt{2},\ 22{+}8\sqrt{2},\ \ldots \\ \frac{Rgeom_k}{\sqrt{2}^k} &\to 3 + 2\sqrt{2} \quad = 5.828427\ldots \qquad \text{A156035} \end{aligned}$$



This $Rgeom$ limit is greater than the corresponding hull boundary length HB from (10). Initially the hull boundary is greater, but eventually the spiralling full right boundary is greater. This occurs at k=13,

$$\begin{aligned} Rgeom_k - HB_k &= \begin{cases} -1,\ -\sqrt{2},\ -2 & \text{if } k = 0 \text{ to } 2 \\ \frac{1}{2}(\sqrt{2}{-}1).\sqrt{2}^k - k\sqrt{2} - 2 + 2\sqrt{2} & \text{if } k \ge 3 \end{cases} \\ &> 0 \qquad \text{iff } k \ge 13 \end{aligned}$$

The rational and $\sqrt{2}$ parts of $Rgeom$ are the total straight and total zig-zag parts of R respectively, which are even and odd $Rpart$ terms respectively.

$$\begin{aligned} Rgeom_k &= RgeomRat_k + \sqrt{2}\,RgeomSqrt_k \\ R_k &= RgeomRat_k + 2\,RgeomSqrt_k \\ RgeomRat_k &= [1,0]\,Rpart_k + 2\sum_{j=0}^{k-1}[1,0]_j\,Rpart_j \\ &= [3,4].2^{\lfloor k/2 \rfloor} - 2 \end{aligned}$$

$$= 1, 2, 4, 6, 10, 14, 22, 30, 46, 62, 94, \ldots \qquad \text{A027383}$$

$$\begin{aligned} \mathit{RgeomSqrt}_k &= [0, \tfrac{1}{2}]\, \mathit{Rpart}_k + 2 \sum_{j=0}^{k-1} [0, \tfrac{1}{2}]_j\, \mathit{Rpart}_j \\ &= [2, 3] . 2^{\lfloor k/2 \rfloor} - k - 2 \\ &= 0, 0, 0, 1,\ \ 2,\ \ 5,\ \ 8, 15, 22, 37, 52, \ldots \qquad {\scriptstyle k \geq 3}\ \text{A077866} \end{aligned}$$

$\mathit{RgeomRat}_k$ is also the right boundary increase between levels,

$$dR_k = R_{k+1} - R_k \quad = \mathit{RgeomRat}_k$$

Unit squares on the right boundary are beside each straight segment and beside each pair of zig-zag segments.

$$\begin{aligned} \mathit{RQpart}_k &= [1, \tfrac{1}{2}]\, \mathit{Rpart}_k \ \ = 2^{\lfloor k/2 \rfloor} - [0, 1] \\ &= 1, 0, 2, 1, 4, 3, 8, 7, 16, 15, 32, \ldots \qquad \text{A106624} \\ \mathit{RQ}_k &= \mathit{RgeomRat}_k + \mathit{RgeomSqrt}_k \\ &= [5, 7] . 2^{\lfloor k/2 \rfloor} - k - 4 \\ &= 1, 2, 4, 7, 12, 19, 30, 45, 68, 99, 146, \ldots \qquad {\scriptstyle k \text{ even}}\ \text{A097809} \end{aligned}$$

The curling ends of the right boundary don't touch so they are at least 1 unit square away from the mid-line and so these RQ squares are distinct (none common to the two inward curls).

The straight and zig-zag part expansions from (11) show that the right boundary segments are all single-traversed and directed from curve start to end. They also give the point number n which is the start of part k. The first part k=0 starts at n=0 and thereafter on expansion that n goes forward or back 1 segment,

$$\begin{aligned} \mathit{PartN}_k &= \mathit{PartN}_{k-1} + [-1, 1] \quad \text{starting } \mathit{PartN}_0 = 0 \\ &= \tfrac{1}{3}\bigl(2^k + [-1, 1]\bigr) \\ &= 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \ldots \qquad \text{A001045} \\ &= \text{binary } 101010\ldots \text{ for } k{-}1 \text{ bits and lowest set to } 1 \end{aligned}$$

The total number of segments in a part is the increment, since the end of part k is the start of part k+1.

$$\begin{aligned} \mathit{PartSegments}_k &= \mathit{PartN}_{k+1} - \mathit{PartN}_k \\ &= 2\, \mathit{PartSegments}_{k-1} + [2, -2] \\ &= \tfrac{1}{3}\bigl(2^k + [2, -2]\bigr) \\ &= 1, 0, 2, 2, 6, 10, 22, 42, 86, 170, 342, \ldots \qquad {\scriptstyle k \geq 1}\ \text{A014113} \end{aligned}$$

3.1 Right Boundary Segment Numbers

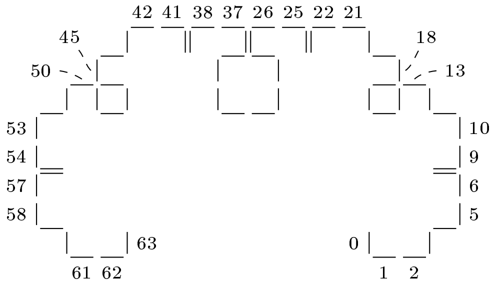


Figure 6:
k=6
right boundary
segment numbers

Theorem 9. *Number the segments of C curve k starting n=0 for the first segment. Write n in binary with k bits. Discard the highest bit run. Then consider pairs 00 or 11.*

$$n = \underbrace{\boxed{H \cdots H}}_{\textit{discard highest run 0s or 1s}}\boxed{\cdots \textit{seek bit pairs 00 or 11} \cdots} \qquad \textit{k bits}$$

Segment n is or is not on the right boundary of the curve according as

$$\mathit{Rpred}_k(n) = \begin{cases} 1 & \textit{if all 00 or 11 pairs are even distances apart} \\ 0 & \textit{if any pairs 00 or 11 are an odd distance apart} \end{cases}$$
$$= \begin{cases} 1 & \textit{pairs 00 and 11 alternate} \\ 0 & \textit{among pairs 00 or 11 any consecutive 00,00 or 11,11} \end{cases}$$

A run of 3 bits 000 *is reckoned as two pairs* 00 *and* 00 *at distance 1 apart which is odd so Rpred* = 0*. Likewise 3 bits* 111*.*

For the curve continued infinitely, infinite 0-bits are considered at the high end and they are the highest run discarded, so the pairs check starts from the highest 1-bit of n.

$$\begin{aligned} \mathit{Rpred}_\infty(n) &= 1,1,1,1,\ 1,1,1,0,\ 0,1,1,1,\ 1,1,0,0,\ 0,0,1,1,\ldots \\ \textit{=1 at} \quad n &= 0,1,2,3,4,5,6,9,10,11,12,13,18,\ldots \\ \textit{=0 at} \quad n &= 7,8,14,15,16,17,23,24,27,28,29,\ldots \end{aligned}$$

As an example, n=715 is binary "0001011001011" in 13 bits for level k=13. The highest run is 000, discard that to 1011001011. There are pairs 11, 00, 11. They are all even distances apart, or equivalently they alternate 11,00, so 715 is on the boundary. On the other hand n=710 binary 1011000110 has pairs 11, 00, 00, 11. The two 00 are an odd distance apart, and equivalently they are consecutive 00,00 (overlapping) so not on the boundary.

$$\mathit{Rpred}_{13}(715) = 1 \qquad\qquad \mathit{Rpred}_{13}(710) = 0$$

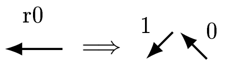
$$715 = \underbrace{0\,0\,0}_{\text{high}}1\,0\,1\,1\,0\,0\,1\,0\,1\,1 \qquad 710 = \underbrace{0\,0\,0}_{\text{high}}1\,0\,1\,1\,0\,0\,0\,1\,1\,0$$

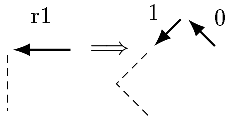
pairs
00 and 11

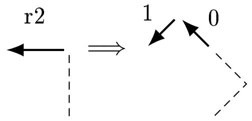
pairs
00 and 11

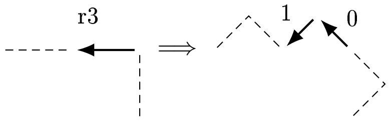
Proof. A given boundary segment may have further segments before or after. Segments expand by at most $\frac{1}{2}$ for each two levels so no other segments can touch the given segment.

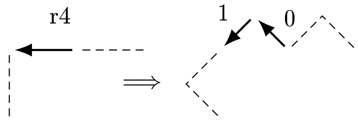
When the segment expands it becomes two segments which are a new low bit 0 or 1 for the segment number and a new configuration of segments before or after. The following configurations $r0$ to $r7$ occur and expand as

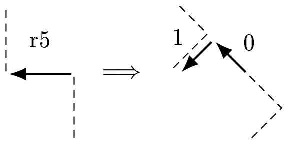


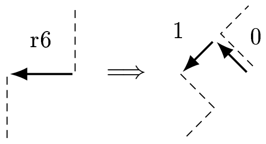


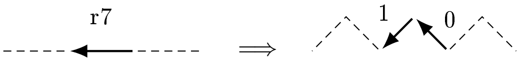








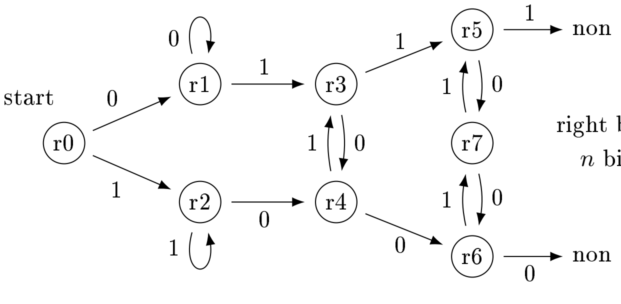




$r0$ is the single initial segment of level k=0. $r1$ and $r2$ are the end-most segments of $k \geq 1$. $r3$ and $r4$ are the right and left end of a straight section. $r7$ is a middle segment in a straight section. $r5$ and $r6$ are the right and left sides of a "V" zig-zag notch.

$r0$ expands as shown to two segments, configuration $r1$ for a new 0-bit or configuration $r2$ for a new 1-bit. Continuing in this way each of the 8 configurations are variously reached.

The transitions between configurations are as follows, taking bits of n high to low. In $r5$ the 1-bit segment is overlapped by the expanded segment after it and so is a non-boundary segment. Similarly in $r6$ the 0-bit segment is overlapped by the segment before and so is a non-boundary segment.



right boundary states,
n bits high to low

A run of high 0-bits is consumed by $r0$ and $r1$. A run of high 1-bits is consumed by $r0$ and $r2$. So $r3$ or $r4$ are reached by the 1 or 0 bit after the high run (respectively). $r3$ and $r4$ alternate between each other for 010101 etc until 11 goes to $r5$ or 00 to $r6$.

In $r5$ alternating bits 010101 go to $r7$ and back, or an 00 goes to $r6$. In $r6$ alternating bits 101010 go to $r7$ and back, or a 11 goes to $r5$. So state $r5$ means that a 11 pair was the last seen of either 00 or 11. And $r6$ means a 00 pair was last seen.

In $r5$ a further 1 is either 111 or with bounces to $r7$ a 11. . . 010101. . . 1. In both cases it is pairs 11 an odd distance apart, and equivalently a 11 pair followed by another 11 pair. For a triplet 111 the two pairs are overlapping. When bouncing to $r7$ they are distinct 11s.

Similarly in $r6$ a further 0 is either 000 or with bounces to $r7$ an 00. . . 101010 . . . 0. In both cases it is pairs 00 an odd distance apart, and equivalently a 00 pair followed by 00 (possibly overlapping). ☐

As noted, a triplet 000 or 111 is treated as pairs 11,11 or 00,00 which are 1 bit apart so non-boundary. So any n with a run of 3 or more consecutive bits is non-boundary for Rpred_∞. (For finite k the highest run is ignored.)

$$\text{any run } {\geq}3 \text{ bits} = 7, 8, 14, 15, 16, 17, 23, 24, 28, 29, \ldots \qquad \text{A136037}$$

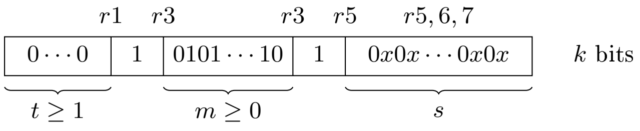
These are a subset of Rpred_∞. The first non-boundary without such a triplet is n=27 “11011”.

$$\begin{aligned} &\mathit{NonRpred}_\infty \text{ without 3 bit 000 or 111} \\ &= 27, 36, 54, 73, 91, 100, 107, 108, 109, 146, 147, 148, 155, \ldots \end{aligned}$$

Total *Rpred* is the right boundary length R_k from theorem 8.

$$R_k = \sum_{n=0}^{2^k-1} \mathit{Rpred}_k(n)$$

This sum can be calculated from the state transitions. Doing so is not as easy as the segment expansions in theorem 8 but is a combinatorial interpretation for the length.



The high t bits go to $r1$ and stay there until 1 to $r3$. Then m optional bounces between $r3$ and $r4$. A 1-bit goes to $r5$ and then must 0-bit to $r7$. At $r7$ there is a choice of $r5$ or $r6$. This is shown as an x which can be either 0-bit or 1-bit.

When an x goes to $r6$ it must be a 1 next back to $r7$ rather than the 0 shown in the sample. In any case it is a no-choice position since after x=1 must have 0 or after x=0 must have 1.

If m is odd then that run ends at $r4$ and the next bit is 0 to $r6$ rather than the 1 shown. The positions of choices and non-choices are the same.

The k digits can be fulfilled by less than all the fields shown. $t{=}k$ all 0s is 1 combination. $t+m = k{-}1$ has the first fixed 1-bit at any of $k{-}1$ positions. $t+m \le k{-}2$ has $s = k - t - m - 2$ ranging 0 to $k{-}2$. The first fixed 1-bit is in any of $k{-}s{-}2$ positions. Within s there are $\lfloor s/2 \rfloor$ bit positions x which can be either 0 or 1.

Starting from a t high run of 1-bits gives corresponding combinations since the states are symmetric in $0{\leftrightarrow}1$ bit flip, so double all the counts.

$$2\left(1 + k{-}1 + \sum_{s=0}^{k-2} (k{-}s{-}2).2^{\lfloor s/2 \rfloor}\right) = R_k \qquad k \ge 1$$

As described in the proof, $r3$ and $r4$ are the start and end segments of a straight section. For the curve continued infinitely r3 is at each $n = \frac{1}{3}(4^j - 1)$ binary "1010...101", and r4 is at each $n = \frac{2}{3}(4^j - 1)$ binary "1010...1010", for integer $j \ge 1$.

These can be seen in the samples of figure 6. $n{=}1, 5, 21$ are the start of straight sections. $n{=}2, 10, 42$ are the ends of those sections. Note that the left side vertical straight starting $n = 53$ is not on the boundary of the curve continued infinitely and is not the $r3_\infty$ bit pattern.

The top middle segments 26 and 37 in figure 6 are each state r7 and to stay on the boundary in their middle position they go to r5 and back to r7, or r6 and back to r7, respectively. These segment numbers are then

$$\begin{aligned}
\mathit{TopMiddlePre}_k &= \begin{cases} 0 & \text{if } k \le 1 \\ \frac{5}{12}\,2^k - [\frac{2}{3}, \frac{1}{3}] & \text{if } k \ge 2 \end{cases} \\
&= \text{binary "11 01 01 \ldots" for } k{-}1 \text{ bits} \\
&= 0, 0, 1, 3, 6, 13, 26, 53, 106, 213, 426, \ldots \qquad {\scriptstyle k \ge 2 \text{ A081254}} \\
\mathit{TopMiddlePost}_k &= \begin{cases} k & \text{if } k \le 1 \\ \frac{7}{12}\,2^k - [\frac{1}{3}, \frac{2}{3}] & \text{if } k \ge 2 \end{cases} \\
&= 2^k{-}1 - \mathit{TopMiddlePre}_k \\
&= \text{binary "100 10 10 \ldots" for } k \text{ bits} \\
&= 0, 1, 2, 4, 9, 18, 37, 74, 149, 298, 597, \ldots \qquad {\scriptstyle k \ge 2 \text{ A081253}}
\end{aligned}$$

Scaled to 0 to 1 for the fractal, the limits are the top middle points in Lévy's big final diagram,

$$\frac{\mathit{TopMiddlePre}_k}{2^k} \to \frac{5}{12} \qquad \frac{\mathit{TopMiddlePost}_k}{2^k} \to \frac{7}{12}$$

4 Left Boundary

Theorem 10. *The left-side boundary length of C curve level k is*

$$L_k = \begin{cases} 2^k & \textit{if } k \le 6 \\ \frac{1}{4}[55, 78].2^{\lfloor \frac{k}{2} \rfloor} + 14k - 130 & \textit{if } k \ge 6 \end{cases} \tag{12}$$

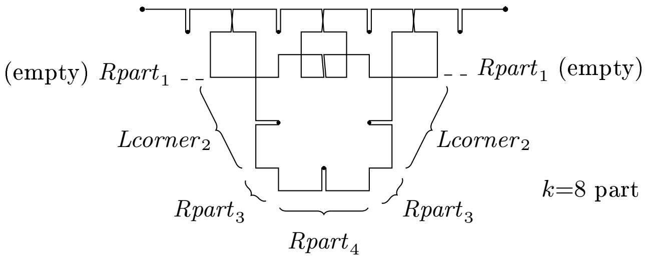
$$= 1, 2, 4, 8, 16, 32, 64, 124, 202, 308, 450, \ldots$$

or a single curve part

$$Lpart_k = \begin{cases} 1, 0, 2, 2, 6, 10 & \textit{if } k = 0 \text{ to } 5 \\ \frac{1}{4}\,[9, 14] . 2^{\left\lfloor \frac{k}{2} \right\rfloor} + 7k - 68 & \textit{if } k \geq 6 \end{cases}$$

$$= 1, 0, 2, 2, 6, 10, 22, 38, 40, 66, 76, \ldots$$

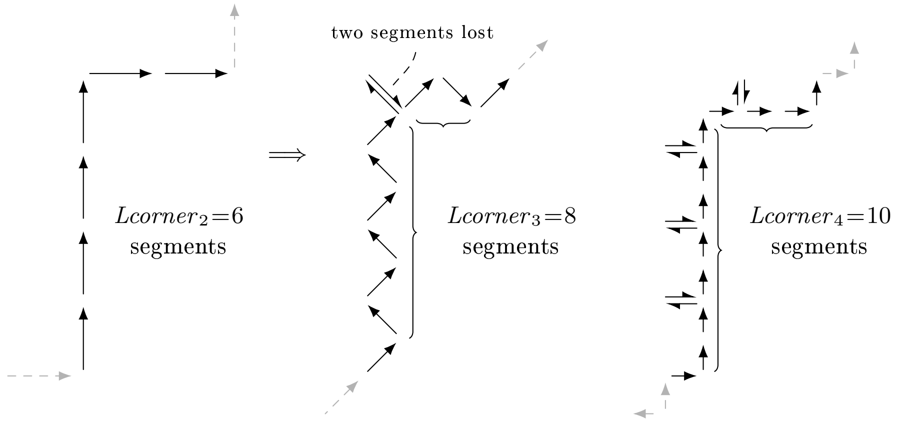
Proof. Each expansion of the right boundary as from above makes pairs of segments on the left side. There is 1 pair between each 2 segments of *Rpart* for k even, so $2^{k/2-1} - 1$. The following diagram shows k=8 with a row of 7 pairs at the top, then a row of 3 pairs which have expanded twice to be 2×2 squares, and finally a single pair expanded 4 times to a big cycle.



The sides expand out at the same rate and so the bottom of each row overlaps its immediate neighbouring rows above and below but no further than that. In all cases the loops go anti-clockwise around.

The pair expanded which is bottom-most has bottom side $Rpart_{k-4}$ and diagonal sides $Rpart_{k-5}$. But the overlap between the rows means the vertical is then not a full *Rpart* side. Let $Lcorner_{k-6}$ be the number of segments in the vertical and horizontal of the overlap corner. $Lcorner_2$ is shown above in part k=8.

Lcorner is outward expanding segments so goes like *Rpart* in extra segments or not at the ends. But the corner loses 2 segments to overlap at odd k.



$$Lcorner_k = \begin{cases} Lcorner_{k-1} + 2 & \text{if } k \text{ even, zig-zag becomes straight} \\ 2\,Lcorner_{k-1} - 4 & \text{if } k \text{ odd, straight becomes zig-zag} \end{cases}$$

$$\text{starting } Lcorner_2 = 6$$

$$= 2.2^{\lceil k/2 \rceil} + [2,0]$$

$$= 6, 8, 10, 16, 18, 32, 34, 64, 66, 128, 130, \ldots \quad k{\geq}2 \qquad \text{2×A228693}$$

For $k \geq 6$ the start and end of the part alternate between two shapes

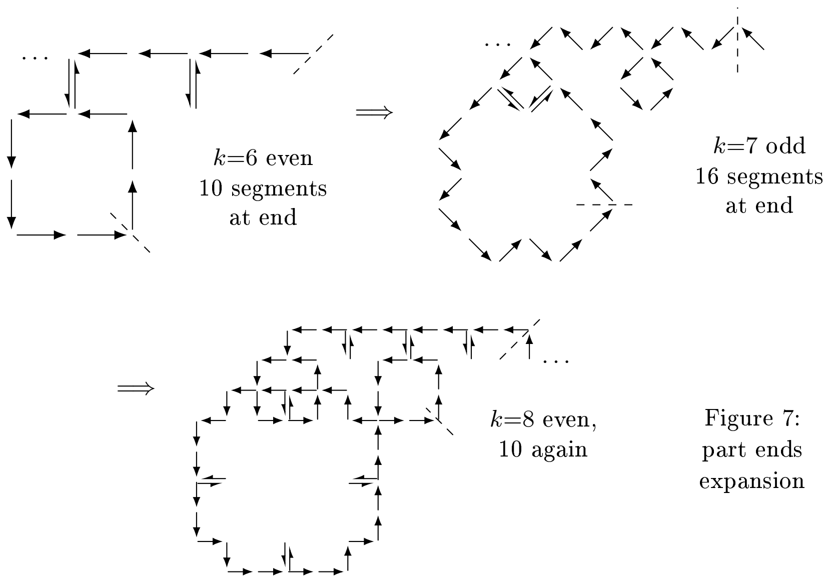


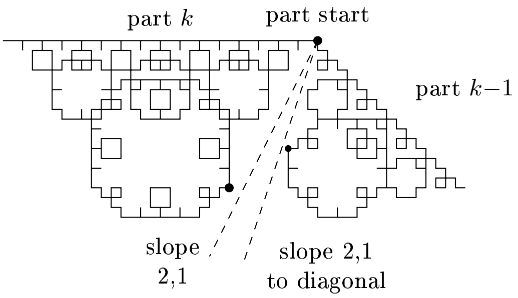
Figure 7:
part ends
expansion

For k even there are 10 segments after then last $Lcorner_2$. For k odd there are 16 segments. This is at each end of the part.

$$\begin{aligned} Lpart_k = {} & Rpart_{k-4} + 2\,Rpart_{k-5} && \text{for } k \geq 6 \\ & + 2\left(Lcorner_{k-6} + Rpart_{k-7}\right) && \text{sum terms} \ldots \\ & + 2\left(Lcorner_{k-8} + Rpart_{k-9}\right) \\ & + \cdots + 2\left(Lcorner_{2\,\mathrm{or}\,3} + Rpart_{1\,\mathrm{or}\,2}\right) \; + 2.[10,16] \end{aligned} \tag{13}$$

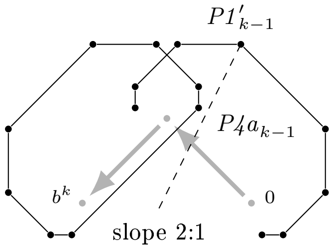
The sum terms (13) go $k{-}6$, $k{-}8$ etc down to $Lcorner_2$ or $Lcorner_3$ according as k even or odd. When k=6 or 7 the sum is taken as empty.

For the full curve the left sides of each curve part do not overlap since they are contained within a line of slope 2;1 relative to the top side.



The middle biggest part shown is even k. On expanding twice to the next k the offset between the dotted vertex and part end doubles, and add i for the part end moving upwards and 1 for the vertex moving left. This is then offset $(-1+2i)2^k + -1-i$ which is a fixed $-1-i$ inside the $-1+2i$ slope. Similarly for odd k on the diagonal.

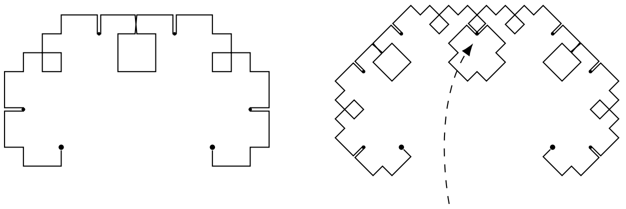
Or in terms of convex hull k formed from two $k-1$, the new loop point shown above is $P4a$ of the second copy of $k-1$. Slope 2:1 line in the first $k-1$ is uncrossed, and the second $k-1$ hull stays within 2:1 for the new middle part k.



Each row of pairs has its endmost expand in the same way as the middle and so all of them have their vertex inside the slope. So the whole left boundary is sum of parts

$$L_k = Lpart_k + 2\sum_{j=0}^{k-1} Lpart_j \qquad \square$$

In L formula (12), for k=6 the two cases 2^k and the subsequent form both give the same $L_6 = 2^6 = 64$. k=6 is the last level which has all segments on the left boundary and after that $L_k < 2^k$ for $k \geq 7$.



k=6 all segments on left boundary $L_6 = 2^6 = 64$

k=7 has 4 segments not on left boundary $L_7 = 2^7 - 4 = 124$

When scaled by $\sqrt{2}^k$ for unit length endoints the odd and even k cases do not converge to the same value, the same as the right boundary segments do not. But if the zig-zag sides are taken geometrically across as distance $\sqrt{2}$ then they do. $\mathit{Lcorner}_k$ is a zig-zag when k odd (like *Rpart*). So geometric lengths become

$$\mathit{LgeomCorner}_k = \begin{cases} \mathit{Lcorner}_k & \text{if } k \text{ even} \\ \tfrac{1}{2}\sqrt{2}\,\mathit{Lcorner}_k & \text{if } k \text{ odd} \end{cases} = 2.\sqrt{2}^k + [2,0]$$

$$\begin{aligned} \mathit{LgeomPart}_k = {} & \mathit{RgeomPart}_{k-4} + 2\,\mathit{RgeomPart}_{k-5} \qquad \text{for } k \geq 6 \\ & + 2\left(\mathit{LgeomCorner}_{k-6} + \mathit{RgeomPart}_{k-7}\right) \\ & + \cdots + 2\left(\mathit{LgeomCorner}_{2\,\mathrm{or}\,3} + \mathit{RgeomPart}_{1\,\mathrm{or}\,2}\right) \; + 2.[10,16] \\ & = \left(\tfrac{5}{4}+\tfrac{1}{2}\sqrt{2}\right)\sqrt{2}^k \; + \left[-2-\sqrt{2},\ 0\right].k \; + \left[2\sqrt{2},\ 28-9\sqrt{2}\right] \end{aligned}$$

$$\begin{aligned} \mathit{Lgeom}_k &= \mathit{LgeomPart}_k + 2\sum_{j=0}^{k-1} \mathit{LgeomPart}_j \\ &= \begin{cases} L_k & k \leq 5 \\ \left(\tfrac{23}{4}+4\sqrt{2}\right)\sqrt{2}^k + \left(1-\tfrac{1}{2}\sqrt{2}\right)k^2 + \left(28-7\sqrt{2}\right)k & \\ \quad + \left[-372+56\sqrt{2}\ - 374+57\sqrt{2}\,\right] & k \geq 6 \end{cases} \\ &= 1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 122+\sqrt{2},\ 192+4\sqrt{2},\ 274+17\sqrt{2},\ \ldots \end{aligned}$$

The segments at the start and end of each part could be treated by some kind of geometric step directly across. But they are a fixed 10 or 16 and so don't change the $\sqrt{2}^k$ term. Scaled by $1/\sqrt{2}^k$ for the endpoints a unit length, the limit is the coefficient on those $\sqrt{2}^k$ terms

$$\begin{aligned} \frac{\mathit{Lgeom}_k}{\sqrt{2}^k} &\to \tfrac{23}{4} + 4\sqrt{2} &&= 11.406854\ldots \\ \frac{\mathit{LgeomPart}_k}{\sqrt{2}^k} &\to \tfrac{5}{4} + \tfrac{1}{2}\sqrt{2} &&= 1.957106\ldots \end{aligned} \tag{14}$$

Part limit (14) is also the ratio of left to right boundary on this geometric measure because $\mathit{RgeomPart}_k/\sqrt{2}^k \to 1$ (the top part being limit 1 as in figure 5). Each part has this same left/right ratio limit and so the total too.

$$\frac{\mathit{Lgeom}_k}{\mathit{Rgeom}_k} \to \frac{\mathit{LgeomPart}_k}{\mathit{RgeomPart}_k} \to \tfrac{5}{4} + \tfrac{1}{2}\sqrt{2} \qquad \text{same as (14)}$$

Left boundary squares follow in a similar way to (13) but with *RQpart* squares instead of segments. For *Lcorner* squares there are 1 fewer squares than segments when k even, or half when odd. That, initial cases, and patterns at start and end of each part gives

$$\begin{aligned} \mathit{LQpart}_k &= \begin{cases} 1,0,2,2,4,6 \\ \tfrac{1}{4}[7,9].2^{\lfloor k/2 \rfloor} \; + [0,3] \end{cases} \\ &= 1,0,2,2,4,6,14,21,28,39,56,\ldots \end{aligned}$$

$$\mathit{LQ}_k = \mathit{Lpart}_k + 2 \; + 2\sum_{j=0}^{k-1} \left(\mathit{LQpart}_j - 1\right) \qquad k \geq 2 \tag{15}$$

$$= \begin{cases} 1, 1, 2, 4, 8, 16 \\ \frac{1}{4}[39, 55].2^{\lfloor k/2 \rfloor} + k - 50 \end{cases}$$
$$= 1, 1, 2, 4, 8, 16, 34, 67, 114, 179, 272, \ldots$$

For sum of parts at (15) there is a unit square common to adjacent parts, hence -1 to count only once. But $LQpart_1 = 0$ so it has no unit squares to overlap, hence $+2$ adjusting for that and the formula applicable only $k \geq 2$.

The total curve boundary length or squares is sum of left and right.

$$\begin{aligned}
B_k &= R_k + L_k \\
&= \begin{cases} 2, 4, 8, 16, 30, 56 & \text{if } k = 0 \text{ to } 5 \\ \frac{1}{4}[83, 118].2^{\lfloor k/2 \rfloor} + 12k - 136 & \text{if } k \geq 6 \end{cases} \\
&= 2, 4, 8, 16, 30, 56, 102, 184, 292, 444, 648, \ldots \\
Bpart_k &= Rpart_k + Lpart_k \\
&= \begin{cases} 2, 0, 4, 4, 10, 16 & \text{if } k = 0 \text{ to } 5 \\ \frac{1}{4}[13, 22].2^{\lfloor k/2 \rfloor} + [4, 8] & \text{if } k \geq 6 \end{cases} \\
&= 2, 0, 4, 4, 10, 16, 30, 52, 56, 96, 108, \ldots \\
BQ_k &= RQ_k + LQ_k \\
&= \begin{cases} 2, 3, 6, 11, 20, 35 & \text{if } k = 0 \text{ to } 5 \\ \frac{1}{4}[59, 83].2^{\lfloor k/2 \rfloor} - 54 & \text{if } k \geq 6 \end{cases} \\
&= 2, 3, 6, 11, 20, 35, 64, 112, 182, 278, 418, \ldots \\
BQpart_k &= RQpart_k + LQpart_k \\
&= \begin{cases} 2, 0, 4, 3, 8, 9 & \text{if } k = 0 \text{ to } 5 \\ \frac{1}{4}[11, 13].2^{\lfloor k/2 \rfloor} + [0, 2] & \text{if } k \geq 6 \end{cases} \\
&= 2, 0, 4, 3, 8, 9, 22, 28, 44, 54, 88, \ldots
\end{aligned}$$

4.1 Left Boundary Segment Numbers

Theorem 11. *The segment numbers n on the left boundary of the C curve are given by the state machine of figure 8 traversed by bits high to low.*

For curve k write n in k many bits (high 0s as necessary). Start at s0 and if ever reach "non" then a non-boundary segment, otherwise boundary.

For the curve continued infinitely, start at state s1 (as if infinite high 0-bits).

$$Lpred_\infty(n) = 0 \quad at\ n = 55, 56, 71, 72,\ 92, 93, 94, 101, 102, 103, \ldots$$

Proof. The boundary segment numbers follow from the pattern of expansions in theorem 10. Total 49 distinct segment types arise, labelled s0 to s38, r3 to r7, and d1,d2, d12, d21.

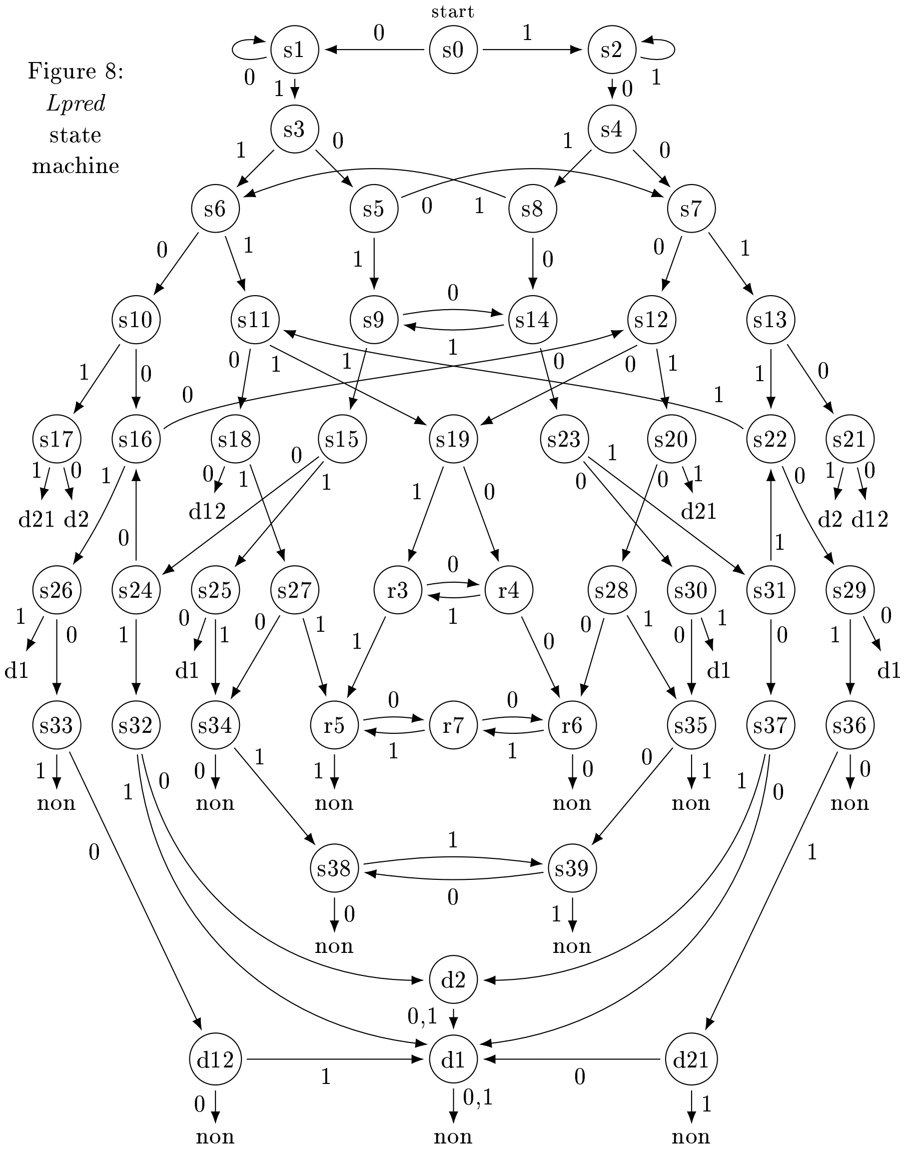
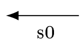
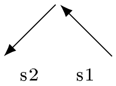


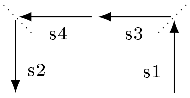
Figure 8:
Lpred
state
machine



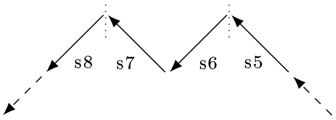
k=0



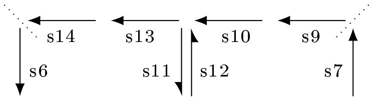
k=1



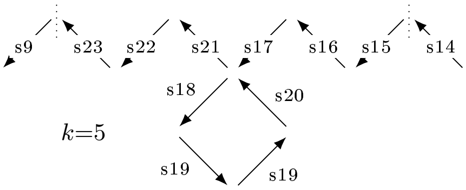
k=2



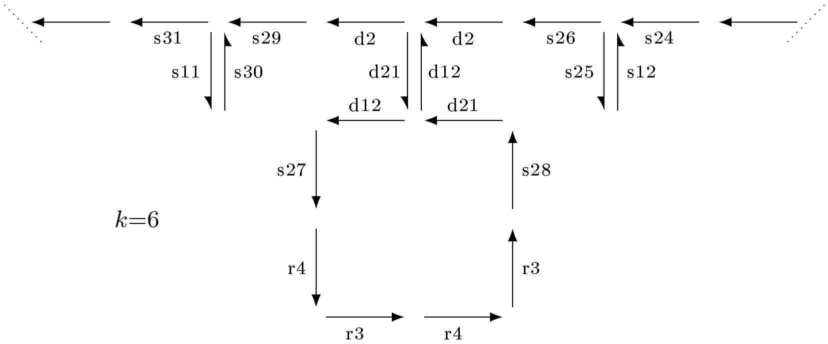
k=3



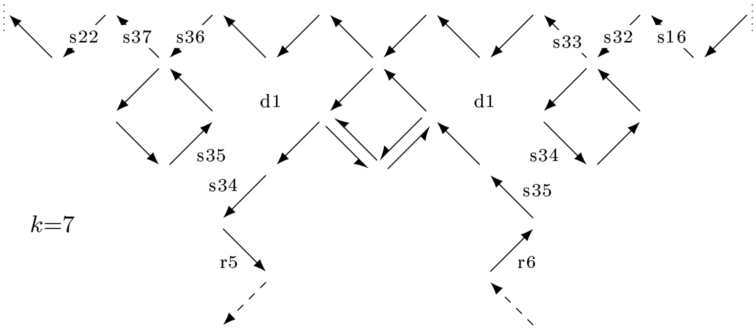
k=4



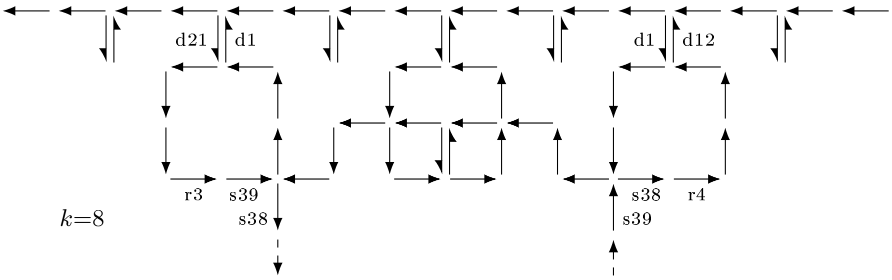
k=5



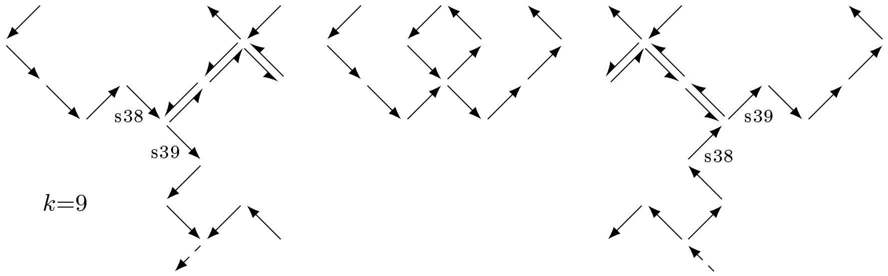
k=6



k=7



k=8



Ends of the curve part are shown dotted. A segment at the end of a part can expand to add a segment to its adjacent part. That new segment is the last or first segment of the adjacent part. For example s9 expands to s14,s15. s14 is the last of part k=4 which precedes in the curve.

When s3 expands it is to a new pair s5,s6. The s5 there is the last of a k=2 part, with a bigger part after it. This differs from s4 which is last segment of a k=2 with smaller part after it. This matters when s5 expands to s7,s9 – it must be s9 in the bigger k=4.

In k=7 at the right, segment type d1 is all 6 segments surrounding there, other than s33 marked. Similarly d1 at the left all except s36. All these d1 segments are enclosed on the next expansion (when s34,s35 touch and close off the regions), for next bit either 0 or 1.

d2 in k=6 is similarly 2 expansions away from enclosed, no matter what next 2 bits. Segment d12 is 1 expansion away for a 0-bit, ie. 0-bit to non, or 2 expansions away for a 1-bit, ie. a 1-bit followed by either 0 or 1. Likewise d21 for opposite 1-bit and 0-bit.

Segments s32,s33 in k=7 are not the same as s17,s21 of k=5 since the left side of s32,s33 is enclosed on the next expansion, whereas s17,s21 is not (being in the middle). Similarly s36,s37 have right side enclosed on next expansion.

The verticals produced by s32,s33 and s36,s37 in k=8 are d1,d12 and d21,d1. Both those segments are reckoned on the boundary.

Segments r3 to r7 are the same as from *Rpred*. They are reached from s19 which is bottom of the k=5 diamond, and also reached by a vertical from each of s18,s20, becoming s27,s28.

In k=8, segments s38,s39 are the 90° corner from theorem 10. When they expand to k=9 the 0-bit of s38 is enclosed and the 1-bit of s39 is enclosed. The resulting zig-zag corner segments in k=9 then expand likewise one enclosed, but the opposite way around. The segment configuration in k=9 is different but the resulting boundary or not on expansion is the same so the segment types are shared. ☐

The s38,s39 corner is a cycle where bit pattern 0101... stays on the boundary. The r5,r6,r7 (an R side in a loop) and r3,r4 (an R start or end in a loop) are also cycles. s9,s14 are start and end of curve parts. States s1,s2 all 0s and all 1s are start and end of the whole curve. Together these are all the cycles in the state machine. Other states eventually reach one of them or non-boundary.

As noted, the verticals d21,d1 and d1,d12 in k=8 are reckoned both segments on the boundary. If considering triangles then the inner d1 could be taken as enclosed already, rather than 1 expansion later. Doing so gives left boundary length L.

$$LpredNin_k(n) = \text{figure 8 but s32 bit 1 and s37 bit 0 to non}$$

$$L_k = \sum_{n=0}^{2^k-1} LpredNin_k(n)$$

5 Area

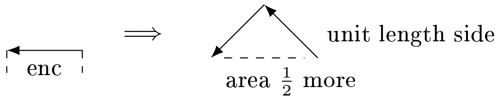
Theorem 12. *The area enclosed by C curve k is*

$$A_k = \begin{cases} 0,0,0,0,0,1 & \textit{if } k = 0 \text{ to } 5 \\ \frac{19}{16}\,2^k - \frac{1}{8}\,[142,201]\,2^{\left\lfloor \frac{k}{2} \right\rfloor} + 2k + 60 & \textit{if } k \geq 6 \end{cases}$$
$$= 0,0,0,0,0,1,6,25,96,284,728,\ldots$$

or a single curve part

$$Apart_k = \begin{cases} 0,0,0,0,0,1 & \textit{if } k = 0 \text{ to } 5 \\ \frac{19}{48}\,2^k - \frac{1}{8}\,[24,35]\,2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{3}\,[8,-2] & \textit{if } k \geq 6 \end{cases}$$
$$= 0,0,0,0,0,1,4,15,56,132,312,\ldots$$

Proof. When the curve expands each existing enclosed unit square increases to side length $\sqrt{2}$ so area doubles. Segments which expand outwards from the boundary of an enclosed area add a further $\frac{1}{2}$ each. A double-traversed segment is segments expanding from an empty area.



Boundary segments which expand inwards would decrease area, but all segments of both left and right boundary expand outwards, except for some at start and end of each curve part. So area increase due to expansion is $\frac{1}{2}Bpart$, less those non-enclosing start and end segments.

From figure 7, for even k there are 4 segments along the top at the end of the curve part which are not on an enclosed area. *Bpart* counts both sides of them and they are the same at both ends of the part, so $\frac{1}{2}Bpart$ is area 8 too big.

For odd k the following diagram shows expansions on each of the boundary segments. Segments on both left and right boundary expand on both sides.

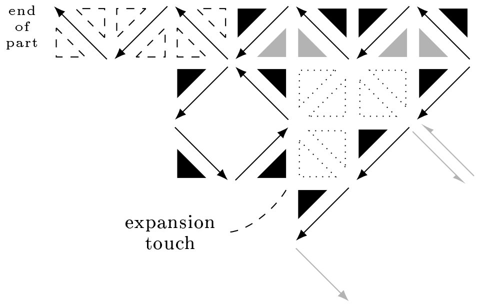


Figure 9:
k odd part,
end area enclosed or
not by expansion of
boundary segments

Black triangles are segment expansions which increase enclosed area. The bottom pair touch and so enclose new area. The black triangles in the top row increase this area.

Gray triangles are on the boundary so are counted by $\frac{1}{2}Bpart$ but are not on the side of expansion. But they are inside the new enclosed area so are wanted.

Dashed triangles at the top left are similarly on the boundary and are counted by $\frac{1}{2}Bpart$, but they are not part of any enclosed area and so should not be counted. But there are a corresponding 6 triangles inside the new enclosed area, shown dotted, which should be counted and are not otherwise. The net result is that no adjustment is needed to $\frac{1}{2}Bpart$ for k odd expanding to k even.

So for $Apart_k$ as a recurrence from the previous $k{-}1$, an adjustment by -8 when k odd which is $k{-}1$ even expanding to k odd.

$$Apart_k = 2\,Apart_{k-1} + \tfrac{1}{2}Bpart_{k-1} - [0,8] \qquad k \geq 7$$

and with initial values through to $Apart_6$ taken explicitly from the curve, the whole area is sum of parts

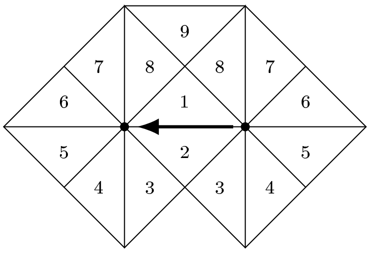
$$A_k = Apart_k + 2\sum_{j=0}^{k-1} Apart_j \qquad \square$$

A measure of density can be made by taking a triangle of area $\frac{1}{4}$ on the right of each segment and comparing that to the enclosed area. The triangles on boundary segments are outside the area but there are at most B_k of them which goes only as a half power of 2 so doesn't change the limit.

$$\frac{\frac{1}{4}2^k}{A_k} \to \frac{4}{19} = 0.210526\ldots \qquad \text{A021479}$$

6 Triangles in Regions

Consider the following regions around the ends of the C curve. The numbering follows Duvall and Keesling[2] but stopping at 9 since the curve is symmetric in horizontal mirror image so the regions on each side are mirror images.



Quarter triangles beside each unit segment as from figure 3 fall variously into these regions. The number in each region can be counted as a density measure.

Theorem 13. *The number of triangles in each region above for curve level k is a recurrence*

$$TR_k = TR_{k-1} + 2\,TR_{k-2} \; - TR_{k-4} + TR_{k-5} \; + 2\,TR_{k-7} + 4\,TR_{k-8} \quad (16)$$

The recurrence is the same for the count in each region, with different initial values.

$$\begin{aligned}
TR(k,1) &= 1,0,0,0,0,0,\ 0,\ 2,\quad 6,\ 10,\ 22,\ 40,\ 80,156,308,622,\ldots\\
TR(k,2) &= 0,0,0,0,0,0,\ 0,\ 2,\quad 2,\ 6,\ 10,\ 20,\ 40,\ 76,156,310,\ldots\\
TR(k,3) &= 0,0,0,0,0,0,\ 1,\ 1,\quad 3,\ 5,\ 10,\ 20,\ 38,\ 78,155,311,\ldots\\
TR(k,4) &= 0,0,0,0,0,1,\ 1,\ 3,\quad 5,\ 10,\ 20,\ 38,\ 78,155,311,\ldots\\
TR(k,5) &= 0,0,0,0,1,1,\ 3,\ 5,\quad 10,\ 20,\ 38,\ 78,155,311,\ldots\\
TR(k,6) &= 0,0,0,1,1,3,\ 5,10,\quad 20,\ 38,\ 78,155,311,\ldots\\
TR(k,7) &= 0,0,1,1,3,5,10,20,\quad 38,\ 78,155,311,\ldots\\
TR(k,8) &= 0,1,1,1,1,2,\ 4,\ 8,\quad 18,\ 39,\ 79,159,315,\ 628,1250,\ldots\\
TR(k,9) &= 0,0,0,2,4,8,16,30,\quad 60,116,232,466,932,1872,3744,\ldots
\end{aligned}$$

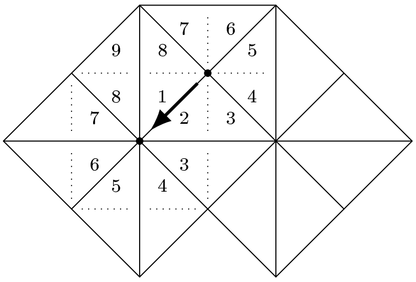
Generating functions

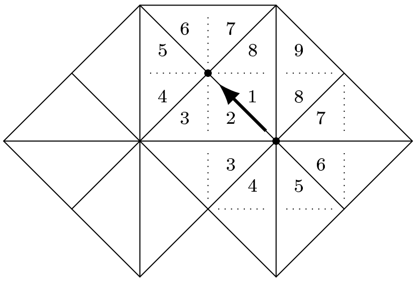
$$\begin{aligned}
gTR1(x) &= \frac{2}{105}.\frac{1}{1{-}2x} + \frac{1}{6}.\frac{1}{1{+}x} + \frac{1}{20}.\frac{7{-}x}{1+x^2} + \frac{1}{28}.\frac{13{+}5x{-}10x^2{-}6x^3}{1-x^2+2x^4}\\
gTR7(x) &= \frac{16}{105}.\frac{1}{1{-}2x} + \frac{1}{12}.\frac{1}{1{+}x} + \frac{1}{40}.\frac{-3{-}x}{1+x^2} + \frac{1}{56}.\frac{-9{-}11x{+}22x^2{+}2x^3}{1-x^2+2x^4}\\
gTR8(x) &= \frac{8}{105}.\frac{1}{1{-}2x} - \frac{1}{12}.\frac{1}{1{+}x} + \frac{1}{40}.\frac{-9{+}7x}{1+x^2} + \frac{1}{56}.\frac{13{+}33x{+}18x^2{-}6x^3}{1-x^2+2x^4}\\
gTR9(x) &= \frac{24}{105}.\frac{1}{1{-}2x} + \frac{1}{10}.\frac{2{-}x}{1+x^2} + \frac{1}{14}.\frac{-6{-}5x{-}4x^2{+}6x^3}{1-x^2+2x^4}
\end{aligned}$$

$$gTR2(x) = 2x.gTR3(x) \quad gTR3(x) = x.gTR4(x) \quad gTR4(x) = x.gTR5(x)$$

$$gTR5(x) = x.gTR6(x) \quad gTR6(x) = x.gTR7(x)$$

Proof. k=0 is a single triangle in region 1. Thereafter a curve level k consists of two level $k{-}1$ curves directed as follows



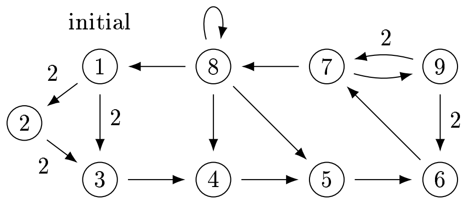


The total triangles in each region are the triangles from the two sub-curves $k{-}1$ which fall in it. For example at the top region 9 has triangles from $k{-}1$ regions 7 and 6 of the left sub-curve and regions 6 and 7 of the right sub-curve, total $2\,TR(k{-}1,6) + 2\,TR(k{-}1,7)$.

$$\begin{aligned}
TR(k,1) &= 2\,TR(k{-}1,2) + 2\,TR(k{-}1,3)\\
TR(k,2) &= 2\,TR(k{-}1,3)\\
TR(k,3) &= \quad TR(k{-}1,4)
\end{aligned}$$

$$\begin{aligned}
TR(k,4) &= TR(k{-}1,5)\\
TR(k,5) &= TR(k{-}1,6)\\
TR(k,6) &= TR(k{-}1,7)\\
TR(k,7) &= TR(k{-}1,8) + TR(k{-}1,9)\\
TR(k,8) &= TR(k{-}1,1) + TR(k{-}1,4) + TR(k{-}1,5) + TR(k{-}1,8)\\
TR(k,9) &= 2\,TR(k{-}1,6) + 2\,TR(k{-}1,7)
\end{aligned}$$

The following diagram shows the relationships in the recurrences. Along the bottom for example $3 \to 4$ is $TR(k,3) = TR(k{-}1,4)$. The chain $2 \to 3 \to 4 \to 5 \to 6 \to 7$ is seen. Copies of some of them are picked out too by 8 and 9. The edges marked 2 are where there is a factor of 2.



Repeated substitution or a little linear algebra give the recurrence (16) for each. The generating functions follow from the recurrence and initial values. ☐

There are 2^k triangles in total. The regions repeated on each side count twice,

$$\begin{aligned}
2^k \;=\;& TR(k,1) + TR(k,2) + TR(k{-}1,9)\\
&+ 2\left(\begin{array}{c} TR(k,3) + TR(k,4) + TR(k,5)\\ +\; TR(k,6) + TR(k,7) + TR(k,8)\end{array}\right)
\end{aligned}$$

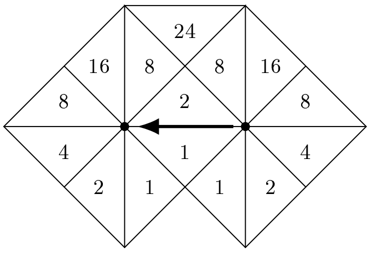
The counts for regions 3 to 7 are identical, just starting each one k later. The count for region 2 is the same too, with a factor of 2. These are a spiralling out by 45° each time.

The recurrence for $TR(k,2)$ starts with a single 2 and other initial values all zeros. Regions 3 to 7 can begin from a single initial 1 similarly if zeros at negative indices are allowed.

The roots of the characteristic polynomial of the recurrence and the generating functions are

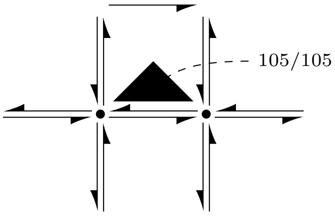
$$\begin{aligned}
& tr_1 = 2, \qquad tr_2 = -1, \qquad tr_{3,4} = \pm i,\\
& tr_{5,6,7,8} = \pm\sqrt{\frac{1 \pm \sqrt{7}i}{2}} = \begin{array}{c} \pm\ \tfrac{1}{2}\sqrt{2\sqrt{2}+1}\\ \pm i\,\tfrac{1}{2}\sqrt{2\sqrt{2}-1}\end{array} = \begin{array}{l} \pm\ 0.978318\ldots\\ \pm i\,0.676096\ldots\end{array} \qquad \tfrac{-1}{2}{+}\text{A190260}\\
& \left|tr_{5,6,7,8}\right| = \sqrt[4]{2} = 1.189207\ldots \qquad \text{A010767}
\end{aligned}$$

The largest root is $tr_1 = 2$ so the coefficients of terms $1/(1{-}2x)$ in the generating functions give the proportion of the total 2^k triangles in each region as $k{\to}\infty$. Those generating functions which are an x factor of another are $\frac{1}{2}$ its coefficient, so successive halvings for $7, 6, 5,$ etc.



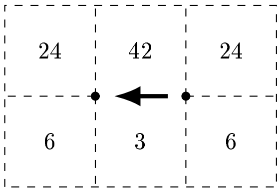
$\frac{1}{105}$ density as $k\to\infty$

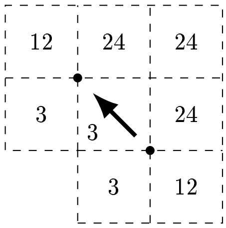
A full 105/105 triangle occurs when there are 15 surrounding curves all contributing their fractions, $24+2\times16+4\times8+2\times4+3\times2+3\times1=105$. The first such full triangle occurs in k=14. Per Duvall and Keesling this is the first triangle region entirely part of the C curve fractal.



One use for these densities could be in computer graphics to approximate the fractal by some gray-scale colouring at the limit of drawing resolution.

If a line segment is the side of a square pixel then that line contributes to 6 surrounding pixels. If a line segment is a diagonal across a pixel then it contributes to 8 surrounding pixels. With the curve endpoints horizontal the two cases are k even or odd.

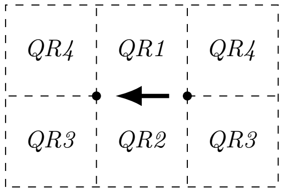




A full-weight side square is 4 full triangles so total 420. A full-weight diagonal square is 2 full triangles so total 210.

In practice the main shape of the curve remains. The gray tends to spread out resulting in a lot of low weight locations. It can help to raise the contrast of those to distinguish them from the background.

For finite k the triangles in square regions are the sum of the respective *TR* parts.



$$\begin{aligned}
QR(k,1) &= TR(k,1) + 2\,TR(k,8) + TR(k,9) \\
QR(k,2) &= TR(k,2) + 2\,TR(k,3) &&= 2\,QR(k{-}2,3) \\
QR(k,3) &= TR(k,4) + TR(k,5) &&= QR(k{-}2,4) \\
QR(k,4) &= TR(k,6) + TR(k,7)
\end{aligned}$$

The TR recurrence is the same for each region so the QR recurrence is the same for each region. In QR the alternating ± 1 term $1/(1{+}x)$ from the TR generating functions cancels out so reducing to an order 7 recurrence.

$$QR_k \;=\; 2\,QR_{k-1} \;-\; QR_{k-4} + 2\,QR_{k-5} - 2\,QR_{k-6} + 4\,QR_{k-7}$$

starting

$$\begin{aligned}
QR(k,1) &= 1,2,2,4,6,12,24,48,102,204,412,824,1642,\ldots \\
QR(k,2) &= 0,0,0,0,0,\;\; 0,\;\; 2,\;\; 4,\;\;\; 8,\;\; 16,\;\; 30,\;\; 60,\;\; 116,\ldots \\
QR(k,3) &= 0,0,0,0,1,\;\; 2,\;\; 4,\;\; 8,\;\; 15,\;\; 30,\;\; 58,116,\;\; 233,\ldots \\
QR(k,4) &= 0,0,1,2,4,\;\; 8,15,30,\;\; 58,116,233,466,\;\; 936,\ldots
\end{aligned}$$

Generating functions

$$\begin{aligned}
\mathit{gQR1}(x) &= \frac{1-2x^2-x^4}{(1-2x)(1+x^2)(1-x^2+2x^4)} \\
&= \frac{2}{5}.\frac{1}{1{-}2x} + \frac{1}{10}.\frac{1{+}2x}{1{+}x^2} + \frac{1}{2}.\frac{1{+}2x}{1-x^2+2x^4} \\
\mathit{gQR2}(x) &= 2x^2\,\mathit{gQR3}(x) \\
\mathit{gQR3}(x) &= x^2\,\mathit{gQR4}(x) \\
\mathit{gQR4}(x) &= \frac{x^2}{(1-2x)(1+x^2)(1-x^2+2x^4)} \\
&= \frac{8}{35}.\frac{1}{1{-}2x} - \frac{1}{20}.\frac{1{+}2x}{1{+}x^2} - \frac{1}{28}.\frac{5{+}10x{-}6x^2{-}12x^3}{1-x^2+2x^4}
\end{aligned}$$

7 Single and Double Segments

Per theorem 1, each segment in the C curve is traversed either once or twice.

The number of single and double traversed segments can be found from the curve pairs and expansions used by Strichartz and Wang[5] for the Hausdorff dimension of the fractal boundary. They show the Hausdorff dimension follows from the largest root of

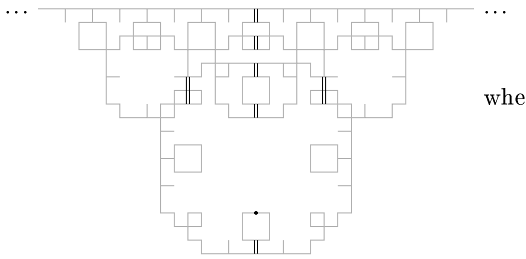
$$\begin{aligned}
&\mathit{poly9}(x) = x^9 - 3x^8 + 3x^7 - 3x^6 + 2x^5 + 4x^4 - 8x^3 + 8x^2 - 16x + 8 \\
&\text{largest root } \; r = 1.954776\ldots \qquad (17) \\
&\mathit{dimH} = \frac{\log\, r}{\log\sqrt{2}} = 1.934007\ldots \qquad \text{A191689}
\end{aligned}$$

r is also the matrix eigenvalue calculated by Duvall and Keesling from full segment configurations (see section 8).

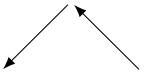
This polynomial occurs in various recurrence characteristic polynomials, and in the generating functions with powers reversed

$$\begin{aligned} \textit{revpoly9}(x) &= 1 - 3x + 3x^2 - 3x^3 + 2x^4 + 4x^5 - 8x^6 + 8x^7 - 16x^8 + 8x^9 \\ &= (1 - rx)(1 - r_2 x) \ldots \end{aligned}$$

To count double-visited segments it's convenient to start from new doubles arising in the join of two k curves at right angles, which is how the curve repeats to form level k+1.



new doubles $DJ_9 = 9$
when two k=9 curves overlap



The new doubles occur only in the new biggest curve part k+1 as this is all which overlaps.

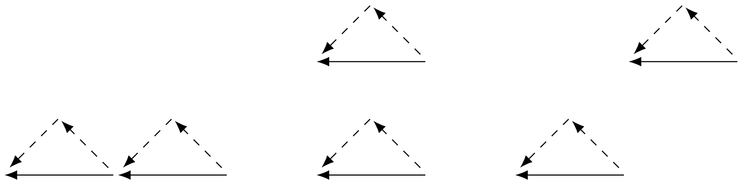
The first double-segment seen in k=4 of figure 1 is where two k=3 curves have joined at +90°. There are no new doubles in k=5, only the replications of those from k=4, since the k=4 join is just a unit square.

Theorem 14. *The number of new double-traversed segments formed by two C curves level k joined at* +90° *is*

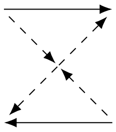
$$\begin{aligned} DJ_k &= 3DJ_{k-1} - 3DJ_{k-2} + 3DJ_{k-3} - 2DJ_{k-4} - 4DJ_{k-5} \qquad k{\ge}9 \qquad (18) \\ &\quad + 8DJ_{k-6} - 8DJ_{k-7} + 16DJ_{k-8} - 8DJ_{k-9} \\ &= 0, 0, 0, 1, 0, 1, 0, 3, 4, 9, 12, 39, 72, 141, 264, 547, 1036, \ldots \end{aligned}$$

$$\textit{Generating function} \quad gDJ(x) = x^3 \, \frac{1 - 3x + 4x^2 - 6x^3 + 8x^4 - 4x^5}{\textit{revpoly9}(x)}$$

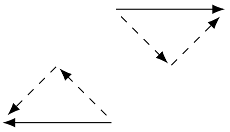
Proof. The overlap between two k curves at a given orientation and offset is determined by the overlaps of their two k−1 sub-curves. The following diagrams are per Strichartz and Wang and show the possible relative positions of pairs of curves. Expansions of the curves are shown dashed.



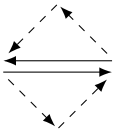
p1 → p3,p3, p7 p2 → p3, p3 p3 → p10



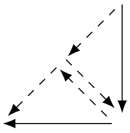
p4 → p7,p7, p12,p12



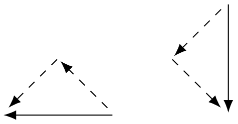
p5 → p4, p10,p10



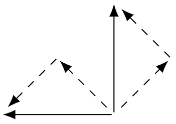
p6 → p11, p11



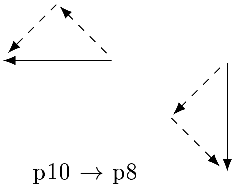
p7 → p1, p6, p9,p9



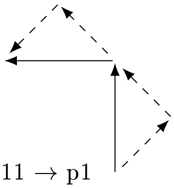
p8 → p5, p8



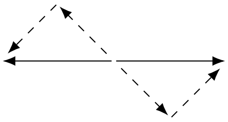
p9 → p2, p9



p10 → p8



p11 → p1

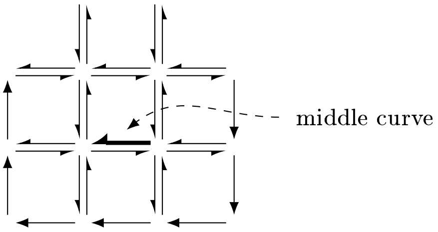


p12 → p12

Two curves at +90° for the theorem are pair p11. New sub-pairings are formed by taking one sub-curve from the first curve and one sub-curve from the second curve. There are four such combinations. For p11 only the middle pairing is close enough to overlap. This is configuration p1. So a p11 level k becomes a p1 level $k-1$.

The configurations with separated segments all have them a unit distance horizontally and/or vertically. For example in p2 the second segment is a unit distance above.

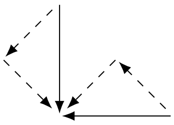
Whether a sub-pair is too far away to overlap is determined by the convex hull around each. The following diagram shows the 30 curve locations which are close enough to have the hull overlapping with the hull of the middle curve.



In p11 the hulls around the two end sub-parts are disjoint, as is the middle from one curve and the end from the other.

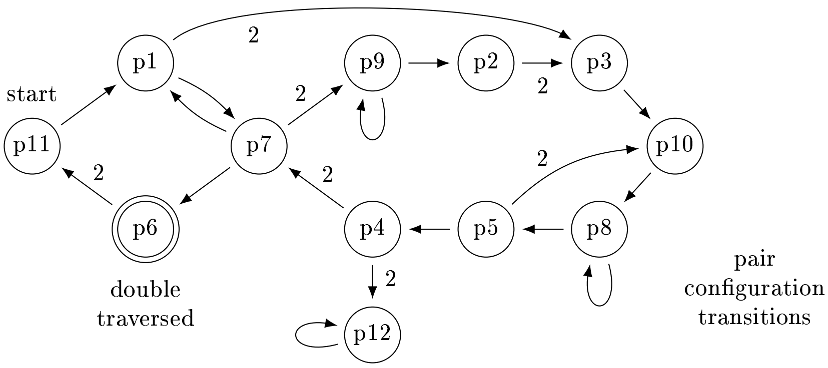
Each configuration is taken to include its mirror image. p3 is shown with the upper segment on the right side and that configuration includes upper segment on the left. p1 expands to one p3 in the orientation shown and one of mirror image.

For vertical segments the side of the expansion is mirrored horizontally too and that means vertical segments reverse. For example p9 mirrors as



The curve expands symmetrically so starting symmetrically the number of occurrences of a configuration and its mirror image are equal and can thus be counted together.

The following diagram shows how the configurations expand. The transitions marked "2" are where there are 2 copies of the new configuration. Other transitions are 1 copy.



The following mutual recurrences are the number of occurrences of each configuration after k expansions. For example $p1_{k+1} = p7_k + p11_k$ because p7 and p11 both expand to p1 in the next level so the number of p1 is how many p7 and p11 there were in the previous level.

$$\begin{aligned}
p1_{k+1} &= p7_k + p11_k & p7_{k+1} &= p1_k + 2p4_k \\
p2_{k+1} &= p9_k & p8_{k+1} &= p8_k + p10_k \\
p3_{k+1} &= 2p1_k + 2p2_k & p9_{k+1} &= 2p7_k + p9_k \\
p4_{k+1} &= p5_k & p10_{k+1} &= p3_k + 2p5_k \\
p5_{k+1} &= p8_k & p11_{k+1} &= 2p6_k \\
p6_{k+1} &= p7_k & p12_{k+1} &= 2p4_k + p12_k
\end{aligned}$$

These equations can be written as a matrix multiply of a column vector of configuration counts. Each row is an equation and represents where the configuration came from. Each column c is where a configuration c goes to.

Strichartz and Wang write the "expands to" in rows, so their M is for a row vector of configuration counts (ie. transposed). p12 is a dead-end and is discarded in their calculation and is not needed for DJ_k here either.

Repeated substitution or a little linear algebra gives recurrences for each count, starting $p11_0$=1 and other initial counts all 0. The double-traversed segments DJ_k is p6. All counts except p12 have the same recurrence as DJ but different initial values. p12 is a cumulative

$$p12_k = 2\sum_{j=0}^{k-1} p4_j \qquad \text{with } p12_0 = 0$$

$$= 0, 0, 0, 0, 0, 0, 0, 4, 8, 16, 40, 92, 176, 344, \ldots$$

The generating function $gDJ(x)$ follows from the recurrence. Or a little polynomial linear algebra $(I - Mx)^{-1}.\mathit{initial}$ gives a column vector of all the generating functions, where M the matrix of the equations, I an identity matrix, and x the polynomial variable. ☐

p4 is the only configuration where all four pairs of its expanded parts are close enough to go to new configurations. All other configurations have some non-overlapping sub-pairs. A non-overlap configuration could be included in the calculation if desired so that each configuration would have exactly 4 outputs and a total $2^k.2^k = 4^k$ pairings at each stage.

Theorem 15. *The number of double-traversed segments in C curve level k is*

$$\begin{aligned} D_k = 2D_{k-1} + DJ_{k-1} \quad &= \sum_{j=0}^{k-1} 2^{k-1-j} DJ_j \qquad (19) \\ = 5D_{k-1} - 9D_{k-2} + 9D_{k-3} - 8D_{k-4} + 16D_{k-6} & \qquad k \geq 10 \\ - 24D_{k-7} + 32D_{k-8} - 40D_{k-9} + 16D_{k-10} & \\ = 0, 0, 0, 0, 1, 2, 5, 10, 23, 50, 109, 230, 499, 1070, \ldots & \end{aligned}$$

Generating function

$$gD(x) = \frac{x}{1-2x}\, gDJ(x) \quad = \tfrac{1}{2}\left(\frac{1}{1-2x} - gS(x)\right)$$

The number of single-traversed segments in the C curve level k is

$$\begin{aligned} S_k &= 2^k - 2D_k \\ &= \textit{same recurrence as } DJ_k \textit{ from } (18)\textit{, but starting} \\ &\quad 1, 2, 4, 8, 14, 28, 54, 108, 210, 412, 806, 1588, 3098, 6052, \ldots \end{aligned}$$

Generating function

$$gS(x) = \frac{1 - x + x^2 - x^3 - 2x^4 + 6x^5 - 4x^6 + 12x^7 - 8x^8}{\mathit{revpoly9}(x)}$$

Proof. DJ_k is the number of new double-traversed segments between two level k curves, hence the recurrence (19) and starting from D_0=0 the sum. The first $gD(x)$ form is from the recurrence written as

$$gD(x) = x\Big(2\, gD(x) + gDJ(x)\Big)$$

and the resulting factor $\frac{x}{1-2x}$ is the usual way to take a sum of descending powers of 2 for a generating function.

D_k counts each double-traversed segment location once, so the total segments are $S_k + 2D_k = 2^k$ and from that S_k, $gS(x)$, and the second $gD(x)$ form. ☐

Total $S + D$ is the number of distinct segments traversed by the curve

$$S_k + D_k = 1, 2, 4, 8, 15, 30, 59, 118, 233, 462, 915, 1818, 3597, 7122, \ldots$$

distinct traversed segments

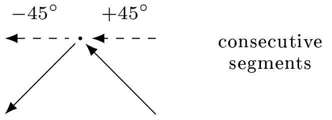
A double-traversed segment is formed from consecutive segments whenever the curve has $\mathit{turn}(n) \equiv 2 \bmod 4$ for 180° reversal. The double in the middle of k=4 is the first of these. Non-consecutive doubles occur when curve sections further apart expand or rotate to touch. The “middle overlap” in k=6 is the first of these.

Consecutive-segment doubles can be counted from the turn formula. They are n with $\mathit{CountLowZeros}(n) \equiv 3 \bmod 4$ for $1 \le n < 2^k$ so a 1-bit then zeros in the low 4 bits or multiple of 4 bits, and any bit values above.

$$\begin{aligned} DC_k &= \sum_{j=4,8,\ldots\le k} 2^{k-j} = \tfrac{1}{15}\left(2^k - 2^{k \bmod 4}\right) && \text{consecutive doubles} \qquad (20) \\ &\quad \text{where } 2^{k \bmod 4} = 2^0, 2^1, 2^2, 2^3 \text{ as } k \equiv 0 \text{ to } 3 \bmod 4 \\ &= 0,0,0,0,1,2,4,8,17,34,68,136,273,546,1092,\ldots && \text{A083593} \\ &= \text{binary } 1000\,1000\,1000\ldots \text{ of } ;k{-}3 \text{ bits} \end{aligned}$$

A consecutive-segment double is part of DJ only when the endpoint of level k at $n = 2^k$ is a 180° turn so that the last segment of the first curve overlaps the first segment of the second curve. This is $\mathit{turn}(2^k) \equiv 2 \bmod 4$ which is $k \equiv 3 \bmod 4$. The DC sum (20) is equivalent to the D sum (19) applied just to those DJ consecutives.

Consecutive-segment doubles go in a period-4 pattern since on expanding the segment pointing towards the point rotates +45° and the segment pointing away rotates −45°, so they cycle $\mathit{p11} \to \mathit{p1} \to \mathit{p7} \to \mathit{p6}$ and coincide every 4 expansions.



Both ends of p6 make rotating pairs like this, as does the cross pair from p7, but not by consecutive segments. The doubles which arise only by such rotational development can be calculated from the ways p11, p1, p7 and p6 transition among themselves.

$$\begin{aligned} \mathit{p1rot}_{k+1} &= \mathit{p7rot}_k + \mathit{p11rot}_k & \qquad \mathit{p7rot}_{k+1} &= \mathit{p1rot}_k \\ \mathit{p6rot}_{k+1} &= \mathit{p7rot}_k & \mathit{p11rot}_{k+1} &= 2\,\mathit{p6rot}_k \end{aligned}$$

Starting from $\mathit{p11rot}_0 = 1$ the number of p6 double-segments arising only from rotations in a k join is

$$\mathit{DJrot}_k = \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{1}{3}\left(2^{\frac{k-1}{2}} - (-1)^{\frac{k-1}{2}}\right) & \text{if } k \text{ odd} \end{cases} \qquad k \text{ odd A001045}$$

$$= 0,0,\ 0,1,\ 0,1,\ 0,3,\ 0,5,\ 0,11,\ 0,21,\ 0,43,\ 0,85,\ldots$$

And the total double-segments solely from rotations in level k is

$$\begin{aligned} Drot_k &= \sum_{j=0}^{k-1} 2^{k-1-j}\, DJrot_j \\ &= \tfrac{1}{10}\,2^k \;-\; \tfrac{1}{6}\,2^{\lceil k/2 \rceil} \;+\; \tfrac{1}{15}\,[1,2,-1,-2] \end{aligned}$$

where [1,2,−1,−2] means the respective value as $k \equiv 0$ to 3 mod 4

$$= 0, 0, 0, 0, 1, 2, 5, 10, 23, 46, 97, 194, 399, 798, 1617, \ldots$$

The largest power in S_k is root r=1.95477... < 2 so $D_k \to \frac{1}{2}2^k$ as $k \to \infty$. The proportion of consecutive doubles is then

$$\frac{DC_k}{D_k} \to \frac{\frac{1}{15}2^k}{\frac{1}{2}2^k} = \frac{2}{15} = 0.13333\ldots \qquad \text{as } k \to \infty \tag{21}$$

and similarly rotational doubles,

$$\frac{Drot_k}{D_k} \to \frac{\frac{3}{30}2^k}{\frac{1}{2}2^k} = \frac{3}{15} = \frac{1}{5}$$

Theorem 16. *The number of double segments exceeds single segments*

$$D_k > S_k \qquad \textit{iff } k \geq 48$$

If each double-traversed segment is counted as 2, for total $2D_k + S_k = 2^k$ segments then those which are part of a double exceed those which are singles,

$$2\,D_k > S_k \qquad \textit{iff } k \geq 30$$

Proof. Seeking $D_k > S_k$ means $\frac{1}{2}(2^k - S_k) > S_k$ so

$$S_k \;<\; \tfrac{1}{3}2^k \tag{22}$$

The S recurrence, which is the DJ recurrence (18), applied repeatedly gives an identity

$$\begin{aligned} S_k \;=\;& 16S_{k-5} + 22S_{k-6} - 2S_{k-7} - 13S_{k-8} - 26S_{k-9} + 20S_{k-10} \\ &+ 128S_{k-11} + 184S_{k-12} + 168S_{k-13} + 8S_{k-14} - 128S_{k-15} \end{aligned} \tag{23}$$

Suppose (22) is true of $k-5$ through $k-15$ inclusive. Then the positive terms of (23) are an upper bound for S_k

$$\begin{aligned} S_k \;<\;& 16\,\tfrac{1}{3}2^{k-5} + 22\,\tfrac{1}{3}2^{k-6} + 20\,\tfrac{1}{3}2^{k-10} + 128\,\tfrac{1}{3}2^{k-11} \\ &+ 184\,\tfrac{1}{3}2^{k-12} + 168\,\tfrac{1}{3}2^{k-13} + 8\,\tfrac{1}{3}2^{k-14} \\ =\;& \tfrac{677}{2048}\,2^k \;<\; \tfrac{1}{3}2^k \end{aligned} \tag{24}$$

It can be verified explicitly that (22) holds for k=48 through k=62, and does not hold below. Then for $k \geq 63$ use the bound (24) so (22) is true of all $k \geq 48$.

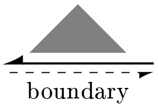
Similarly for $2D_k > S_k$ which becomes $S_k \leq \frac{1}{2}2^k$, first true at the 15 values starting $k = 30$ and thereafter by substituting $\frac{1}{2}2^k$ into (23). ☐

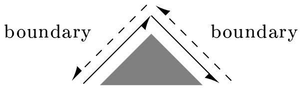
Identity (23) was found by a computer search rolling the S recurrence down until reaching a form where the positive terms are small enough for the powers

2^{-n} in (24) to sum $< \frac{1}{3}$.

Another approach is that difference $D_k - S_k$ (or factor $\frac{3}{2}$ for similar $\frac{1}{3}2^k - S_k$) is a linear recurrence and asking when >0 is a linear recurrence positivity problem. In this case a straightforward one since there is a single largest root 2. So write as coefficients on powers of 2 and the other roots r etc then some computer calculation can find when the power k will certainly be enough to overcome possible negatives. Identity (23) has the attraction of doing the equivalent just in rationals.

Single-traversed segments can count the boundary length of the triangles form of the curve.





A triangle on a single-traversed segment has its long side on the boundary since there is no triangle in the opposite direction. On expanding the curve one level segments are on the short sides of those level k triangles. The single-traversed segments of level k+1 are therefore triangle short sides on the boundary in k. So the number of boundary sides of triangles

$$\begin{aligned} \mathit{ThreeB}_k &= S_k + S_{k+1} \\ &= 3, 6, 12, 22, 42, 82, 162, 318, 622, 1218, \ldots \end{aligned}$$

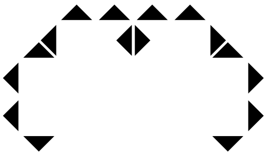


Figure 10: k=4 triangle sides on the boundary
$\mathit{ThreeB}_4 = 42$
$\mathit{ThreeNonB}_4 = 6$

This boundary length relates the growth r^k in S to the Hausdorff dimension of the fractal boundary. The triangle short sides could be taken as their geometric length $\frac{1}{2}\sqrt{2}$ if preferred, so $S_k + \frac{1}{2}\sqrt{2}S_{k+1}$. That still grows as power r^k.

Non-boundary sides of the triangles can be counted similarly by double-traversed segments. Each double segment is 2 non-boundary triangle long sides and at the next level each double segment is 2 non-boundary short sides.

$$\begin{aligned} \mathit{ThreeNonB}_k &= 2\,D_k + 2\,D_{k+1} \\ &= 0, 0, 0, 2, 6, 14, 30, 66, 146, 318, \ldots \end{aligned}$$

In k=4 figure 10 above there are 2 triangles with long sides meeting and 2 places with 2 short sides meeting for $\mathit{ThreeNonB}_6 = 6$.

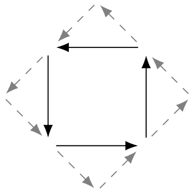
Total boundary and non-boundary are the 3 sides each of 2^k triangles

$$\mathit{ThreeB}_k + \mathit{ThreeNonB}_k = 3.2^k$$

A given triangle can have 1, 2 or 3 boundary or enclosed sides. See the second part of section 8 for counts of triangles of each type.

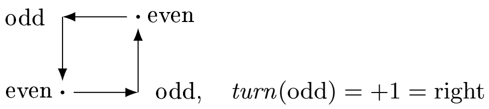
7.1 Outward Squares

The counts of single and double segments give the number of unit squares with outward facing segments, meaning segments anti-clockwise around a unit square so the square is on their left.



Unit square with outward expanding segments

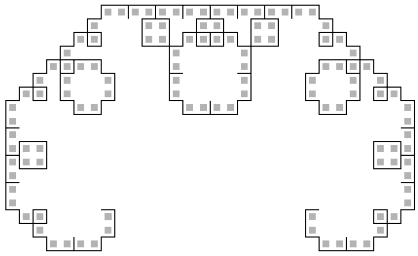
In such a square two of the corners are odd, meaning location $x{+}iy$ has $x{+}y$ odd, and so an odd n along the curve. As from subsection 1.2, always have $turn(n){=}1$ left at odd n so the two segments at the odd corners are consecutive in the curve.



So except for the single segment $k{=}0$, a unit square has either 2 or 4 outward segments. The pairs of consecutive segments have expanded from the previous level as segments across the diagonal. So the 2 or 4 side segments correspond to a single or double traversed segment in the previous level.

$$QO2_k = \begin{cases} 0 & \text{if } k{=}0 \\ S_{k-1} & \text{if } k \geq 1 \end{cases} \qquad \text{squares with outward segments}$$

$$QO4_k = \begin{cases} 0 & \text{if } k{=}0 \\ D_{k-1} & \text{if } k \geq 1 \end{cases}$$



$k{=}8$
distinct unit squares
on left of segments
$QO_8 = S_7 + D_7 = 118$

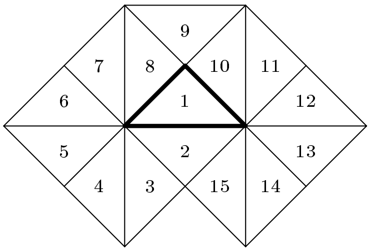
There are a few outward unit squares on the left of the curve at the start and end of curve parts, and otherwise they are inside enclosed areas.

The right boundary segments all go forward along the curve so there are none on the right boundary. The enclosed areas are the same curling around so the boundary is their right.

Some of the $QO4$ unit squares are made by 4 consecutive segments taking 3 consecutive left turns around the square. These are the runs of 3 left turns from subsection 1.2. They are also the expansion of a consecutive double-traversed segment from DC above. The proportion of 4 side consecutive squares within $QO4$ is thus the same as $DC/D \to \frac{2}{15}$ from (21).

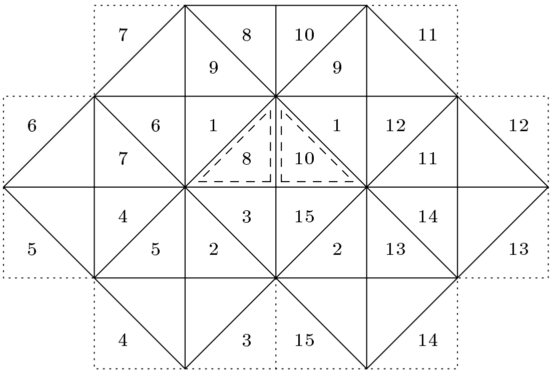
8 Triangle Configurations

Duvall and Keesling[2] show that the contents of a given triangular area adjacent to a line segment on repeated expansion are determined by the contents of itself and 14 neighbours, for total 15 triangles. The following diagram is their numbering.



This set of triangles arises from the curve extents (convex hull). Repeatedly expanding each of the triangles eventually puts sub-triangles into region 1. Any triangle further away than these 15 does not expand into region 1.

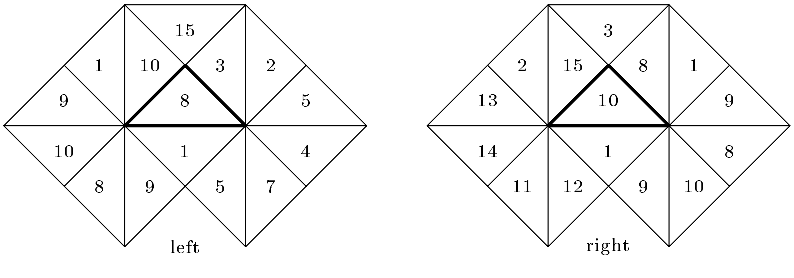
A given configuration has each triangle either filled or unfilled. On expansion each triangle divides in half and its contents spread out as follows.



The original location 1 is the two dashed triangles shown. The expansion makes the left contain whatever was in the original 8, and has a set of other half-triangles surrounding. The right dashed triangle contains whatever was in the original 10, and has a set of other half-triangles surrounding.

It can be noted there are some blank half-triangles in the expansion. They would be from other triangles further away. The expansion of the 15 triangles given is enough to cover the similar 15 half triangles at and around the left and right dashed locations, nothing further away is needed.

The new left and right configurations are



This means that an original arrangement of 15 triangles each filled or unfilled becomes these two new arrangements with filled or unfilled copied from the respective numbered locations.

Note that this expansion follows the location, not the content. This is why neither left nor right have original triangle 1 contents in the middle. The content of 1 has spread out of that location in the expansion, that location instead receiving the content from 8 and 10.

The complete curve is represented by an initial set of 15 configurations which are a single filled triangle in each possible position. The curve begins with a single triangle which is a single line segment and these arrangements are all the locations close enough to have it in one of the 15 parts.

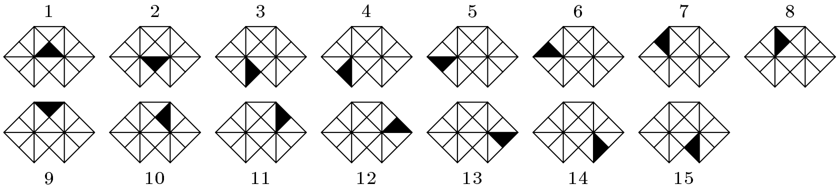


Figure 11: initial configurations

By some computer calculation, 753 distinct non-empty configurations occur from repeatedly expanding these. Duvall and Keesling reduce this to a 734×734 matrix. They note that if all combinations of filled and unfilled occurred then there would be $2^{15}-1 = 32767$ configurations, but only 753 actually occur and at k=19 all have been seen.

Curve properties limit possible configurations. A simple condition is that connected curve means there must be the same number of segments entering as leaving each point, other than curve start and end. Demanding this of the left and right points (the horizontal ends of triangle 1) is 2518 configurations.

$$\text{count}(1,3,5,7) = \text{count}(2,4,6,8) \qquad \text{left point} \tag{25}$$

$$\text{count}(2,10,12,14) = \text{count}(1,11,13,15) \qquad \text{right point} \tag{26}$$

All turns are the same at a given location, as from page 5. So each entering segment present or absent must have the same present or absent as its turned leaving. Turn is left at odd locations ($x+iy$ with $x+y$ odd), so one of the left or right configuration points must be turn left. The other point can be any turn, including another left. Demanding this or curve start/end is the 753 non-empty configurations which occur.

These conditions say nothing about triangle 9, allowing it to be filled or unfilled. The configurations which occur have both 9 filled and 9 unfilled forms, other than the single segments and the 2-segment curve start and ends.

Some configurations are mirror images of another. Since the curve is symmetric the number of occurrences of a configuration and its mirror image are equal. If mirror images are combined then 393 non-empty configurations occur.

When a triangle expands its two half triangles touch at their ends. Initial configurations 1 and 2 in figure 11 have no triangle at either end and therefore do not occur as an expansion, only as initial configurations.

Initial configurations 3 to 8 and 10 to 15 have one end of their triangle with no other touching triangle. This means they are at the start or end of the curve and so occur exactly 1 each in all curve levels. (The remaining single 9 occurs in many places.)

Counts of how many times each configuration occurs can be written as mutual recurrences. A given configuration in curve $k{+}1$ arises as expansion of some configuration from k, possibly several different configurations there. With some computer linear algebra these mutual recurrences can be turned into individual recurrences for count of a given configuration. The characteristic polynomials of these recurrences have $\mathit{poly9}(x)$ from section 7 and further polynomials

$$\begin{aligned}
&\mathit{poly8alt}(x) = x^8 - 2x^6 - x^4 + 2x^2 - 4 \quad = \big((x^2{-}\tfrac{1}{2})^2 - \tfrac{5}{4}\big)^2 - 5 \\
&\qquad \text{largest roots } \ \pm\sqrt{\tfrac{1}{2} + \sqrt{\tfrac{5}{4} + \sqrt{5}}} \ = \pm 1.538538\ldots \\
&\mathit{poly8}(x) = x^8 - x^7 + x^6 - 2x^5 + x^4 - x^3 + \ x^2 \ - 2 \\
&\qquad \text{largest root } \ 1.320638\ldots \\
&\mathit{poly11}(x){=}x^{11}{-}2x^{10}{+}3x^9{-}5x^8{+}6x^7{-}9x^6{+}10x^5{-}14x^4{+}12x^3{-}14x^2{+}8x{-}8 \\
&\qquad \text{largest root } \ 1.642339\ldots \\
&\mathit{poly14}(x) = x^{14} - 2x^{10} - 5x^6 - 6x^4 - 4x^2 - 8 \\
&\qquad \text{largest roots } \ \pm 1.438110 \\
&\mathit{poly18}(x) = x^{18}{-}5x^{17}{+}12x^{16}{-}21x^{15}{+}28x^{14}{-}25x^{13}{+}8x^{12}{+}19x^{11}{-}53x^{10} \\
&\qquad\quad + 90x^9{-}112x^8{+}118x^7{-}108x^6{+}88x^5{-}48x^4{+}16x^3{+}8x^2{-}32x{+}32 \\
&\qquad \text{largest root } \ r_{18} = 1.848349\ldots
\end{aligned} \tag{27}$$

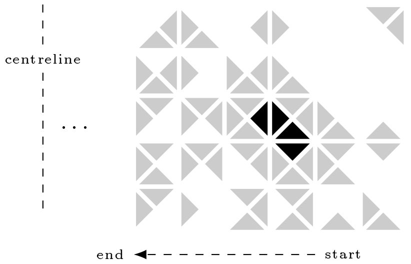
Triangle 1 fully enclosed is the configuration with all 15 triangles filled. It has filled triangles on all its sides and in all directions at its corners. The number of such fully enclosed triangles in curve k is

$$\mathit{Enc}_k = 0,\ldots,0 \text{ (14 zeros)},\ \ 8, 44, 172, 554, 1656, 4714, \ldots$$

given by a 94-term recurrence with characteristic polynomial

$$\begin{aligned}
\mathit{EncPoly}(x) = {} & (x{+}1)\,(x{-}1)^2\,(x{-}2)\,(x^2{+}1)\,(x^4{+}1)^3\,(x^4{-}2)^2 \\
& .\,\mathit{poly8}(x)\,.\,\mathit{poly8alt}(x)\,.\,\mathit{poly9}(x)\,.\,\mathit{poly11}(x)\,.\,\mathit{poly14}(x)\,.\,\mathit{poly18}(x)
\end{aligned}$$

Per Duvall and Keesling, the first fully enclosed triangles occur in $k{=}14$ and there are $\mathit{Enc}_{14} = 8$ of them there. These are 4 adjacent triangles, repeated in mirror image at the other end of the curve.



k=14
fully enclosed triangles,
same in mirror image
at left of curve,
$Enc_{14} = 8$

Each fully enclosed triangle expands to two fully enclosed triangles in the next level, hence term $x-2$ in $EncPoly(x)$ which is power 2^k. From its generating function the coefficient on that power is 1.

$$Enc_k = 2^k - NotEnc_k$$

No other configuration has $x-2$ in its characteristic polynomial so Enc_k is the only one growing as 2^k.

There are 2^k triangles total in the curve so the remaining $NotEnc_k$ is triangles not fully enclosed but rather with one or more side or corner on the boundary. This is what Duvall and Keesling use to establish the Hausdorff dimension of the fractal boundary. The largest root in $NotEnc$ is the largest of Enc other than 2, and this is r=1.954... from $poly9(x)$.

Apart from the 1-segment initial configurations noted above and 4 further 2-segment configurations at curve start or end, all configurations have a count which is a linear recurrence of length 88 or more. The characteristic polynomials all have the six $poly8$ through $poly18$ then various further terms up to degree 4. They all have $poly9$ so grow as power of its root r.

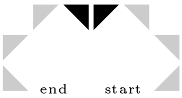
Another form of enclosure can be made by considering how many sides of a given triangle have neighbours. The configurations can count triangles with 1, 2 or 3 sides enclosed by taking triangle 1 filled and counting how many of its neighbouring $2, 8, 10$ are filled. The number of triangles with 0 to 3 such neighbours are

$$\begin{aligned}
Three0_k &= 1,2,4,6,10,18,36,68,126,234,448,844,\ldots\\
Three1_k &= 0,0,0,2,\ \ 6,14,26,54,114,238,476,960,\ldots\\
Three2_k &= 0,0,0,0,\ \ 0,\ \ 0,\ \ 2,\ \ 6,\ \ 16,\ \ 40,\ \ 98,234,\ldots\\
Three3_k &= 0,\ldots\text{(10 zeros)},\ 2,10,36,120,350,\ldots
\end{aligned}$$

$Three0,1,2$ are recurrences of 64 terms with the same characteristic polynomial. $Three3$ has an extra factor $x-2$.

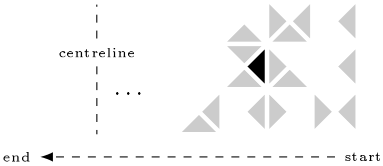
$$\begin{aligned}
&PolyThree0(x) = PolyThree1(x) = PolyThree2(x)\\
&\qquad = (x^2+1)\,.\,(x^4+1)^3\,.\,(x^4-2)\,.\,poly8alt(x)\,.\,poly9(x)\,.\,poly11(x)\,.\,poly18(x)\\
&PolyThree3(x) = (x-2)\,.\,PolyThree0(x)
\end{aligned}$$

For $k = 0,1,2$ all triangles have no neighbours on their sides. For k=3 the triangles in the middle have a side in common.



k=3, triangles with 1 neighbour
$\mathit{Three1}_3 = 2$

The first 3-side enclosed triangle is in k=10 where $\mathit{Three3}_{10} = 2$ is a single triangle, repeated in mirror image at the other end of the curve.



k=10, triangle with 3 neighbours
and mirror image so $\mathit{Three3}_{10} = 2$

The total of all three types is the 2^k curve triangles

$$\mathit{Three0}_k + \mathit{Three1}_k + \mathit{Three2}_k + \mathit{Three3}_k \ = 2^k$$

ThreeB and *ThreeNonB* from section 7 are the total number of boundary and non-boundary sides. *Three0*,*1*,*2*,*3* have respectively 0,1,2,3 non-boundary sides, or 3,2,1,0 boundary sides. On summing those multiples the recurrences or generating functions of *Three0*,*1*,*2*,*3* variously cancel to leave just *poly9* which is in *ThreeB* and *ThreeNonB*.

$$\begin{aligned} \mathit{ThreeB}_k &= 3\,\mathit{Three0}_k + 2\,\mathit{Three1}_k + \mathit{Three2}_k \\ \mathit{ThreeNonB}_k &= \qquad\qquad\qquad \mathit{Three1}_k \ + 2\,\mathit{Three2}_k + 3\,\mathit{Three3}_k \end{aligned}$$

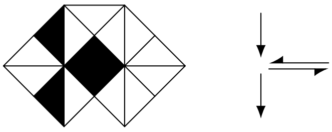
Single and double traversed segments from theorem 15 can be counted by configurations too. Single-traversed segments are locations 1 filled and 2 unfilled. 147 such configurations occur (or 78 with mirror images combined). The total count of all these is S_k. The opposite 2 filled and 1 unfilled is the same result.

Double-traversed segments are locations with both 1 and 2 filled. 224 such configurations occur (or 116 with mirror images combined). There is such a location on both sides of a double segment so the total count of these configurations is $2D_k$.

Configuration counts cover all locations with at least one filled triangle so single and double segments can also be counted by any back-to-back pair of triangles in the configuration, for example 3,4. This has the effect of counting locations which are beside a single or double segment rather than on it. Mirror image configurations are not equivalent in this case.

As noted above the counts for each individual configuration are recurrences of 88 terms or more. But cancellations leave only *poly9* (and 2^k for the doubles) in the total.

As an example of a configuration in the total, 1,2,4,7 filled is the first double segment at the top of k=4 (as from figure 1), here turned +90° so the double is 1,2.



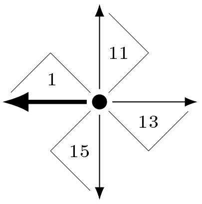
first double segment configuration
(one of 224 such double
configurations altogether)

The count of this configuration in curve level k grows as r^k,

$$\begin{aligned} &\mathit{Tee}_k = 0,0,0,0,1,2,4,6,13,22,40,64,119,205,\ldots && (28)\\ &\mathit{TeePoly}(x) = \text{same as } \mathit{EncPoly}(x) \text{ but change } (x{-}2) \text{ to a second } (x{+}1) \end{aligned}$$

8.1 Visited Points and Inward Squares

The C curve visits points up to 4 times. The number of points which are visited 1 through 4 times can be counted from the configurations. Take triangle 1 filled and then count how many configurations have $11,13,15$ filled. This is the number of segments leaving the point at the right hand end of triangle 1.



Triangle 1 filled
and some of $11,13,15$

The end of the curve is only a destination, so is not included in these counts of segment starts. It is a further 1-visited point.

When a point is visited $v{=}2$ or more times, triangle 1 is any of the segments leaving the point, so the total counts of such configurations are v times the number of v-visited points.

The number of points visited 1 to 4 times are then

$$\begin{aligned} \mathit{V1}_k &= 2,3,5,9,15,27,49,93,171,321,601,1137,\ldots\\ \mathit{V2}_k &= 0,0,0,0,\ \ 1,\ \ 3,\ \ 8,18,\ \ 43,\ \ 93,200,\ \ 414,\ldots\\ \mathit{V3}_k &= 0\ldots(9 \text{ zeros}),\ \ 2,8,28,84,238,596,\ldots\\ \mathit{V4}_k &= 0\ldots(12 \text{ zeros}),\ \ 1,5,24,78,232,626,1648,\ldots \end{aligned}$$

The first double-visited point $\mathit{V2}_4 = 1$ is the top middle in $k{=}4$ (figure 1).

Multiplied by the number of visits the total is the $2^k{+}1$ curve points.

$$\mathit{V1}_k + 2\,\mathit{V2}_k + 3\,\mathit{V3}_k + 4\,\mathit{V4}_k \;=\; 2^k+1$$

The visit counts are recurrences of 35, 18, 35 and 36 terms respectively with characteristic polynomials

$$\begin{aligned} \mathit{PolyV1}(x) = \mathit{PolyV3}(x) &= \qquad\quad \mathit{poly8alt}(x)\,.\,\mathit{poly9}(x)\,.\,\mathit{poly18}(x)\\ \mathit{PolyV2}(x) &= (x{-}1)\,.\,\mathit{poly8alt}(x)\,.\,\mathit{poly9}(x)\\ \mathit{PolyV4}(x) &= (x{-}2)\,.\,\mathit{poly8alt}(x)\,.\,\mathit{poly9}(x)\,.\,\mathit{poly18}(x) \end{aligned}$$

Factor $(x{-}2)$ in $\mathit{PolyV4}(x)$ is power 2^k and all the others grow only as the *poly9* power r^k. From its generating function, the 2^k term in $\mathit{V4}$ is

$$\mathit{V4}_k = \tfrac{1}{4}2^k\ +\cdots$$

V1 and *V3* have the same *poly9* term in their generating function so although *V3* starts slowly their ratio has limit

$$\frac{\mathit{V1}_k}{\mathit{V3}_k} \to 1 \qquad \text{as } k{\to}\infty$$

This is approached from above since $\mathit{V1}_k > \mathit{V3}_k$ always. This can be seen by considering their difference

$$\begin{aligned} \mathit{V1sub3}_k &= \mathit{V1}_k - \mathit{V3}_k \\ &= 2, 3, 5, 9, 15, 27, 49, 93, 171, 319, 593, 1109, \ldots \\ \mathit{PolyV1sub3}(x) &= \mathit{poly18}(x) \end{aligned} \tag{29}$$

This is a recurrence with characteristic polynomial $\mathit{poly18}(x)$ since their other parts cancel. Showing it >0 is a linear recurrence positivity problem. In this case an easy one since there is a single largest root r_{18} (27). Writing *V1sub3* in powers of the roots of *poly18* and suitable factors f,

$$\begin{aligned} \mathit{V1sub3}_k &= f_1\, r_1^k + \cdots + f_{17}\, r_{17}^k\ + f_{18}\, r_{18}^k \\ \frac{\mathit{V1sub3}_k}{r_{18}^k} &= f_1 \Big(\frac{r_1}{r_{18}}\Big)^k + \cdots + f_{17} \Big(\frac{r_{17}}{r_{18}}\Big)^k\ + f_{18} \\ f_{18} &= 1.319372\ldots \end{aligned}$$

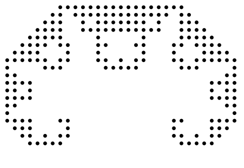
Terms other than f_{18} may be negative but are at worst their magnitude negative and each $|r_j/r_{18}| < 1$ so they decrease with k. Some computer calculation shows those total magnitudes $< f_{18}$ for $k \geq 1$.

The limit for *V2* is smaller than *V1* and *V3*. Writing its *poly9* generating function part in terms of shifts of the corresponding part of *V1* or *V3* (which are the same) gives a ratio limit, using the *poly9* root r (17),

$$\begin{aligned} \frac{\mathit{V2}_k}{\mathit{V1}_k} \to \frac{\mathit{V2}_k}{\mathit{V3}_k} &\to -\tfrac{34}{15} + \tfrac{1}{12} r - \tfrac{1}{10} r^2 + \tfrac{43}{120} r^3 + \tfrac{1}{4} r^4 - \tfrac{31}{240} r^5 + \tfrac{19}{80} r^6 - \tfrac{3}{16} r^7 + \tfrac{3}{80} r^8 \\ &= 0.950615\ldots \end{aligned}$$

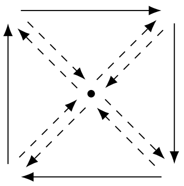
The total of all visited point types is the number of distinct points in the curve. This is a 37 term recurrence

$$\begin{aligned} V_k &= \mathit{V1}_k + \mathit{V2}_k + \mathit{V3}_k + \mathit{V4}_k \\ &= 2, 3, 5, 9, 16, 30, 57, 111, 214, 416, 809, 1579, \ldots \\ \mathit{PolyV}(x) &= (x{-}1)\,(x{-}2)\,.\,\mathit{poly8alt}(x)\,.\,\mathit{poly9}(x)\,.\,\mathit{poly18}(x) \end{aligned}$$



k=8
distinct visited points
$V_8 = 214$

The various visited point counts can also be calculated from unit squares with inward facing segments, meaning segments going clockwise around a unit square so the square is on their right and so they will expand into it.



unit square sides
expanding inward

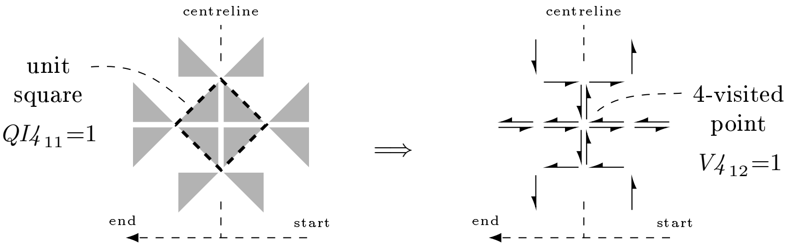
The number of unit squares with 1 to 4 inward sides follow from the configurations

$$\begin{aligned}
\mathit{QI1}_k &= 1, 2, 4, 6, 12, 22, 44, 78, 150, 280, 536, 998, \ldots \\
\mathit{QI2}_k &= 0, 0, 0, 1, 2, 5, 10, 25, 50, 107, 214, 439, \ldots \\
\mathit{QI3}_k &= 0, \ldots \text{ (8 zeros)}, \;\; 2, 6, 20, 56, 154, 358, 848, \ldots \\
\mathit{QI4}_k &= 0 \ldots \text{ (11 zeros)}, \;\; 1, 4, 19, 54, 154, 394, 1022, 2492, \ldots
\end{aligned}$$

These are the increments of the respective *V1* etc. When the curve expands, its existing visited points keep the same visit count and the curve expands into adjacent unit squares to visit a new point in the middle of each square. The number of visits to that point is the number of sides of the square so

$$\mathit{V1}_{k+1} = \mathit{V1}_k + \mathit{QI1}_k \qquad \text{etc} \tag{30}$$

The first 4-inward square occurs in k=11 as single $\mathit{QI4}_{11}$=1 and this becomes the first 4-visited point $\mathit{V4}_{12}$=1. This is located on the centreline between the curve endpoints. The curve is symmetric so it must be on the centreline as anywhere else would have another in mirror image. The following diagram has the curve start to end horizontal which means the unit square in k=11 is at 45°.



first *QI4* inward unit square becoming first *V4* visited point

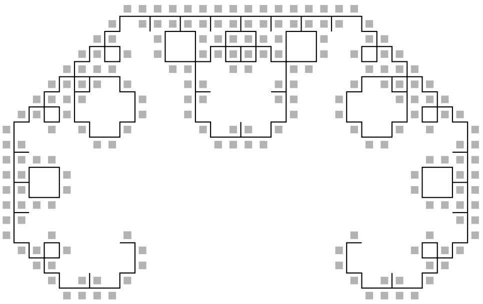
Each segment of the curve has a unit square on its right side so the squares multiplied by number of sides is the 2^k total curve segments.

$$\mathit{QI1}_k + 2\,\mathit{QI2}_k + 3\,\mathit{QI3}_k + 4\,\mathit{QI4}_k \;=\; 2^k$$

Total squares (without multiplying) is the number of distinct unit squares on the right of the curve.

$$\begin{aligned}
\mathit{QI}_k &= \mathit{QI1}_k + \mathit{QI2}_k + \mathit{QI3}_k + \mathit{QI4}_k \\
&= 1, 2, 4, 7, 14, 27, 54, 103, 202, 393, 770, 1494, \ldots
\end{aligned}$$

The curve curls around over itself so these right sides can be on either left or right curve boundary or inside an enclosed area.



k=8
unit squares on
right of segments
$QI_8 = 202$

QI2 squares occur in 2 types. The 2 sides can be adjacent around the square or can be opposite sides of the square.

The number of squares of each type are 17-term recurrences the same as *QI2* but different initial values.

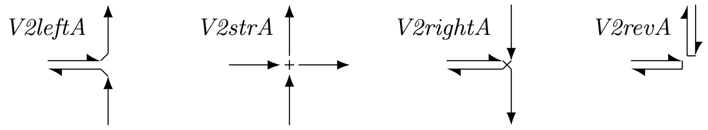
$$QI2o_k = 0,0,0,1,2,5,10,23,46,97,190,383,\dots$$
$$QI2o_k = 0,0,0,0,0,0,\ \ 0,\ \ 2,\ \ 4,10,\ \ 24,\ \ 56,\dots$$

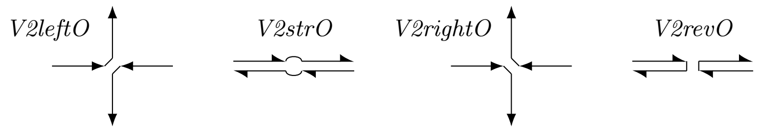
The first opposites $QI2o_7 = 2$ are the squares at each end of part k=7 from figure 9 (which on the next expansion touch to enclose new area).

Both counts grow as the root r. Writing the $poly9(x)$ parts of their generating functions in terms of the corresponding part of *QI2* gives limits for their proportions as various terms in r. The proportion is just under 6 times more adjacent than opposite.

$$\frac{QI2a_k}{QI2_k} \to \frac{\begin{array}{c}3061140 - 624884r + 1337706r^2 - 506076r^3 - 411025r^4\\ +801257r^5 - 675875r^6 + 454961r^7 - 139808r^8\end{array}}{2.2.31.20177} \tag{31}$$
$$= fQI2a \ = 0.856584\dots$$
$$\frac{QI2o_k}{QI2_k} \to 1 - fQI2a \ = 0.143415\dots$$
$$\frac{QI2a_k}{QI2o_k} \to \frac{fQI2a}{1 - fQI2a} \ = 5.972726\dots \qquad \text{as } k{\to}\infty$$

The *QI2a*, *QI2o* squares expand to 2-visit points with left turns. They have either opposing or adjacent segments. Subsequent expansions of those points are then likewise straight, right or reverse turns with opposing or adjacent segments.





Each expansion opens the turn up by 90° decrease, in the manner of turn figure 4. These expansions cycle around to reach the original left again after 4 levels.

$$\begin{tikzcd} \mathit{V2leftA} \arrow[r] & \mathit{V2strA} \arrow[r] & \mathit{V2rightA} \arrow[r] & \mathit{V2revA} \arrow[lll, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \mathit{V2leftO} \arrow[r] & \mathit{V2strO} \arrow[r] & \mathit{V2rightO} \arrow[r] & \mathit{V2revO} \arrow[lll, bend left] \end{tikzcd}$$

So recurrences

$$\begin{aligned} \mathit{V2leftA}_k &= \mathit{QI2a}_k + \mathit{V2leftA}_{k-4} \\ &= 0,0,0,0,1,2,5,10,24,48,102,200,407,794,1591,\ldots \\ \mathit{V2leftO}_k &= \mathit{QI2o}_k + \mathit{V2leftO}_{k-4} \\ &= 0,0,0,0,0,0,0,0,2,4,10,24,58,112,238,\ldots \end{aligned} \tag{32}$$

The other types arise only from the respective turn expansion since they are $\mathit{turn} \equiv 0,-1,-2 \bmod 4$ so not an odd location which is a new expansion point.

$$\begin{aligned} \mathit{V2strA}_k &= \mathit{V2leftA}_{k-1} & \qquad \mathit{V2strO}_k &= \mathit{V2leftO}_{k-1} \\ \mathit{V2rightA}_k &= \mathit{V2leftA}_{k-2} & \mathit{V2rightO}_k &= \mathit{V2leftO}_{k-2} \\ \mathit{V2revA}_k &= \mathit{V2leftA}_{k-3} & \mathit{V2revO}_k &= \mathit{V2leftO}_{k-3} \end{aligned}$$

The total is all 2-visit points

$$\begin{aligned} \mathit{V2}_k = \quad & \mathit{V2leftA} + \mathit{V2strA} + \mathit{V2rightA} + \mathit{V2revA} \\ & + \mathit{V2leftO} + \mathit{V2strO} + \mathit{V2rightO} + \mathit{V2revO} \end{aligned}$$

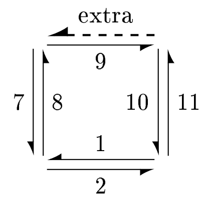
Configurations can count *V2leftA* by suitable triangle combination, for example around the left point 1,2,4,7 present and 3,5,6,8 absent. Similarly the other adjacent *V2* types.

Configurations can count sum $\mathit{V2leftO} + \mathit{V2rightO}$, but not the two separately since the configurations do not distinguish segments consecutive in the curve or not. That information could be added, for more configuration types, but it's enough to go from *QI2o* and how the points expand. Similarly sum $\mathit{V2strO} + \mathit{V2revO}$ together.

Working through generating functions shows limits for proportions of each pair *V2leftA*/*V2leftO* etc are the same as the *QI2a*/*QI2o* at (31).

8.2 Other Squares

Unit squares with a segment of either direction on their sides can be counted by configurations.



The extra segment above 9 is not in the configurations, but is determined by the segments which are. Per outward squares in subsection 7.1, the curve turns mean that for $k{\ge}1$ there are 0, 2 or 4 outward segments around a square. So $\mathit{count}(2,7,11)$ odd or even determines the extra segment present or absent respectively.

The count of sides is then

$$\begin{aligned}\mathit{Qsides}(\mathit{conf}) &= (1\,\text{or}\,2)\ + (7\,\text{or}\,8)\ + (10\,\text{or}\,11)\ + (9\ \text{or odd}\ 2,7,11)\\ &\quad\text{for}\ k \ge 1\end{aligned}$$

The number of unit squares with 1 to 4 segment sides is then

$$\begin{aligned}\mathit{Q1}_k &= 2,2,4,6,10,16,32,54,\ \ 98,166,312,552,\dots\\ \mathit{Q2}_k &= 0,1,2,5,10,17,30,60,116,222,410,789,\dots\\ \mathit{Q3}_k &= 0,0,0,0,\ \ 0,\ \ 2,\ \ 6,14,\ \ 32,\ \ 74,166,350,\dots\\ \mathit{Q4}_k &= 0,0,0,0,\ \ 0,\ \ 1,\ \ 2,\ \ 5,\ \ 10,\ \ 23,\ \ 50,114,\dots\end{aligned}$$

For $k{=}0$ there is a single segment and two 1-sided squares (each side of it). Thereafter $\mathit{Q1},\mathit{2},\mathit{3}$ are recurrences of 80 terms and $\mathit{Q4}$ of 81 terms, with characteristic polynomials

$$\begin{aligned}&\mathit{PolyQ1}(x) = \mathit{PolyQ2}(x) = \mathit{PolyQ3}(x)\\ &\quad = (x{-}1)\,(x{+}1)\,(x^2{+}1)\,(x^4{-}2)\,(x^4{+}1)^3\\ &\qquad .\mathit{poly8alt}(x)\,.\mathit{poly9}(x)\,.\mathit{poly11}(x)\,.\mathit{poly14}(x)\,.\mathit{poly18}(x)\\ &\mathit{PolyQ4}(x) = (x{-}2)\,.\mathit{PolyQ1}(x)\end{aligned} \tag{33}$$

For k even the counts are even since the centreline is segments and the squares on each side of it are in mirror image. For k odd the centreline is squares and the symmetry means those squares cannot be 1 or 3 sides. So $\mathit{Q1}$ and $\mathit{Q3}$ always even, and from the recurrences $\mathit{Q2}$ and $\mathit{Q4}$ parity in a repeating pattern after initial $k = 1,5$.

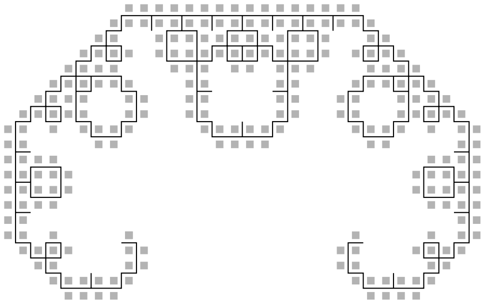
$$\begin{aligned}&\mathit{Q2}_k \ \equiv\ \mathit{Q4}_{k+4}\ \bmod 2\\ &\begin{array}{c}\mathit{Q4}_k\\ \bmod 2\end{array} \ \equiv \begin{cases}0 & \text{if } k{=}1\\ 1 & \text{if } k{=}5\\ [0,1,0,0,\ 0,0,0,1] & \text{otherwise} \qquad \scriptstyle k>5\ \text{and offset A173858}\end{cases}\end{aligned}$$

Number of squares multiplied by number of sides is the segments traversed by the curve, but counted twice since there is a square on each side.

$$\mathit{Q1}_k + 2\,\mathit{Q2}_k + 3\,\mathit{Q3}_k + 4\,\mathit{Q4}_k \ = \ 2\,(S_k + D_k)$$

Total without weighting is the number of unit squares beside the curve.

$$\begin{aligned}Q_k &= Q1_k + Q2_k + Q3_k + Q4_k \\ &= 2, 3, 6, 11, 20, 36, 70, 133, 256, 485, 938, 1805, \ldots\end{aligned}$$



k=8
unit squares
beside the curve
$Q_8 = 256$

Some areas enclosed by the curve are $Q4$ unit squares and some are bigger areas made of multiple squares. The first bigger area is the 2×2 square in k=6 (figure 2). The number of non-unit-square areas can be calculated using Euler's formula for a connected planar graph

$$\textit{vertices} + \textit{inside regions} - \textit{edges} = 1 \tag{34}$$

Vertices are the curve points V_k. Edges are the line segments between them S_k+D_k. Regions are $Q4_k$ squares and $NonQ_k$ non-unit-square areas. So (34) gives

$$\begin{aligned}NonQ_k &= S_k + D_k - V_k - Q4_k + 1 \qquad \text{non unit square areas} \\ &= 0, 0, 0, 0, 0, 0, 1, 3, 10, 24, 57, 126, \ldots\end{aligned}$$

The 2^k parts of D, V, $Q4$ cancel leaving $NonQ$ the same recurrence as $Q1,2,3$ at (33) but different initial values.

The total number of enclosed regions of both kinds is

$$Acount_k = Q4_k + NonQ_k = 0, 0, 0, 0, 0, 1, 3, 8, 20, 47, 107, 240, \ldots$$

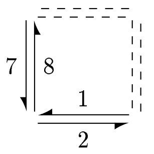
The total area of these is A from section 5. So the mean size of an enclosed area (either unit square or not) has limit

$$\frac{A_k}{Acount_k} \to 4+\tfrac{3}{4} \qquad \text{mean enclosed area size} \tag{35}$$

$NonQ$ grows only as r^k so (35) is simply the coefficients of 2^k in A and $Q4$, being $\frac{19}{16}/\frac{1}{4}$. The $Q4$ contribution to the area is only its $\frac{1}{4}2^k$, count, leaving area $\frac{15}{16}2^k$ from $NonQ$. So although the number of $NonQ$ areas grows slowly, their sizes successively increase so as to contribute the majority of A.

A yet further variation on squares can be made by considering those which have only double-traversed segments as sides (no singles).

In the configurations this is asking that 1=2, 7=8, 10=11, 9=extra. As from subsection 7.1, the outward facing segments are always even so there can be only 2 or 4 sided double squares. The 2-sided double squares have adjacent sides since 2 outward segments must be consecutive.



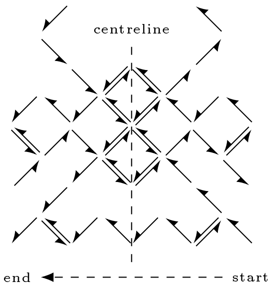
2 sides both double-traversed
always adjacent

$$QD2_k = 0\ldots \text{ (9 zeros)},\ \ 2, 4, 10, 24, 58, 112, 240, 484,$$
$$QD4_k = 0\ldots \text{ (13 zeros)},\ \ 2, 10, 40, 122, 350, 936, 2480, \ldots$$

QD2 is a recurrence of 27 terms with the following characteristic polynomials. *QD4* is a recurrence with the same characteristic polynomial as *Q4* but different initial values.

$$PolyQD2(x) = (x^2{+}1)\,(x^4{+}1)\,(x^4{-}2)\,.\,poly8alt(x)\,.\,poly9(x)$$
$$PolyQD4(x) = PolyQ4(x)$$

The first 4-sided double squares $QD4_{13}{=}2$ occur together on the curve centreline.



$k{=}13$
first unit squares with
4 double-traversed sides
$QD4_{13} = 2$

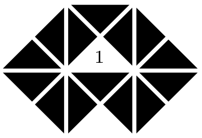
8.3 Triangle Polygon Pieces

Triangles beside curve segments touch at their sides to form polygon pieces. The number of such pieces can be counted.

Lemma 1. *The polygon pieces of the C curve do not contain fully enclosed holes. A fully enclosed hole means a hole which cannot be reached by passing end-to-end through unfilled triangles.*

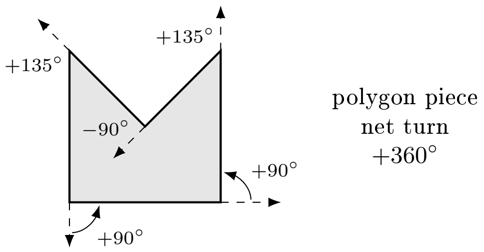
Proof. As from the grid of theorem 1, an infinite set of C curves traverses every segment exactly twice. If there was a hole inside a fully enclosed polygon in any curve then it could not be reached from the outside by another to fill the plane. ☐

The smallest example of a fully enclosed hole not occurring would be triangle 1 unfilled and everything around it filled. Its left point has 3 segments entering but 4 leaving, and vice versa at the right, which is contrary to (25),(26).



empty triangle 1
fully enclosed
does not occur

At a polygon vertex consider the turn there as angle + or − to go to the next side anti-clockwise around. For any polygon the net turn all the way around is 360°.



By lemma 1 there are no holes inside a polygon piece so all triangle boundary sides (*ThreeB*) are on the outside of polygon pieces.

At a triangle end vertex there are one or more polygon pieces. The total turns of the pieces there are reduced by each triangle. No triangles would be each piece reversing $+\frac{1}{2}$ of 360°, then less $\frac{1}{8}$ for each triangle. It doesn't matter how the triangles are distributed among the pieces, just the total.

At a triangle top vertex, in a unit square, similarly each piece reversing $+\frac{1}{2}$ of 360° but net turn reduced $\frac{1}{4}$ for each triangle, which is each side of the square.

$$\mathit{Trigons}_k = \sum_{\text{triangle ends}} \left(\tfrac{1}{2}\mathit{pieces} - \tfrac{1}{8}\mathit{triangles}\right) + \sum_{\text{squares}} \left(\tfrac{1}{2}\mathit{pieces} - \tfrac{1}{4}\mathit{sides}\right)$$

Each polygon vertex has one triangle boundary side anti-clockwise around, so total number of pieces at points is *ThreeB*. Each triangle end is a *V1*,*2*,*3* visited point and there are 2 triangles for each visit, except only 1 at curve start and end. Each triangle top is the side of a *QI1*,*2*,*3* inward square. So

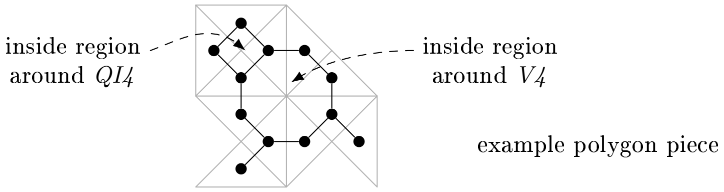
$$\begin{aligned}\mathit{Trigons}_k &= \tfrac{1}{2}\,\mathit{ThreeB}_k - \tfrac{1}{8}.2\big(\mathit{V1}_k + 2\,\mathit{V2}_k + 3\,\mathit{V3}_k - 1\big)\\ &\qquad\qquad - \tfrac{1}{4}\big(\mathit{QI1}_k + 2\mathit{QI2}_k + 3\mathit{QI3}_k - 1\big)\\ &= \tfrac{1}{2}\,\mathit{ThreeB}_k - \tfrac{1}{4}\left(\mathit{V1}_{k+1} + 2\,\mathit{V2}_{k+1} + 3\,\mathit{V3}_{k+1} - 1\right) \qquad (36)\\ &= 1, 2, 4, 7, 13, 25, 49, 95, 183, 353, 685, 1320, \ldots\\ \mathit{TrigonsPoly}(x) &= \mathit{poly8alt}(x)\,.\,\mathit{poly9}(x)\,.\,\mathit{poly18}(x)\end{aligned}$$

(36) uses $\mathit{V1}_{k+1} = \mathit{V1}_k + \mathit{QI1}_k$ etc increment as from (30).

The same sort of calculation can be made from the full triangle configurations by net polygon piece turns at the left point and top point of triangle 1. A given location in the curve has configurations with triangle 1 as any of the 4 rotated positions around it. Duplicates can be eliminated by requiring triangle 1 filled and divide by the number of filled triangles which could be 1 by rotation. For the left end this means the filled among 1,3,5,7, and for the top point 1,8,9,10.

Another approach is to apply Euler's formula (34). Take each triangle as a graph vertex, total 2^k. Edges are between triangles with a side in common, which is total $\frac{1}{2}\mathit{ThreeNonB}_k$. Each polygon piece is a connected part of this graph. Different polygon pieces are not connected.

Inside regions of the graph are around a *V4* enclosed point at the end of a triangle or around a *QI4* inward square around the top of a triangle. By lemma 1 these are the only inside regions of the graph.

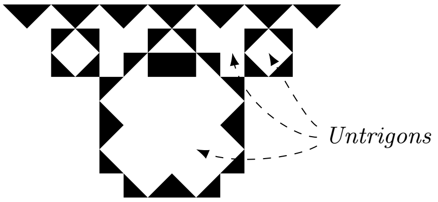


Summing over vertices plus inside regions less edges, for aggregate (34), is 1 for each connected polygon piece so

$$\begin{aligned} \mathit{Trigons}_k &= 2^k + \mathit{V4}_k + \mathit{QI4}_k - \tfrac{1}{2}\,\mathit{ThreeNonB}_k \qquad (37) \\ &= 2^k + \mathit{V4}_{k+1} - \tfrac{1}{2}\,\mathit{ThreeNonB}_k \end{aligned}$$

$\mathit{V4}_{k+1} = \mathit{V4}_k + \mathit{QI4}_k$ is again from (30), giving all regions. The 2^k term here and the 2^k parts of $\mathit{V4}$ and $\mathit{ThreeNonB}$ cancel to leave (36).

The polygon pieces touch at various corners and there are empty "unpolygons" between them.



The number of these unpolygons can be counted using Euler's formula again. Take vertices as the non-enclosed triangle vertices, which is $\mathit{V1{,}2{,}3}_{k+1}$. Edges are the ThreeB sides between such points. Regions are then polygon pieces plus unpolygon pieces.

$$\begin{aligned} \mathit{Trigons}_k + \mathit{Untrigons}_k &= \mathit{ThreeB}_k - \big(\mathit{V1}_{k+1} + \mathit{V2}_{k+1} + \mathit{V3}_{k+1}\big) + 1 \\ \mathit{Untrigons}_k &= \tfrac{1}{2}\,\mathit{ThreeB}_k - \tfrac{1}{4}\big(3\,\mathit{V1}_{k+1} + 2\,\mathit{V2}_{k+1} + \mathit{V3}_{k+1} - 3\big) \qquad (38) \\ &= 0, 0, 0, 0, 0, 1, 3, 10, 24, 57, 131, 295, \ldots \end{aligned}$$

Or Euler's formula again but for unpolygons directly. Take vertices as the points of each triangle, edges as their sides, and regions as all triangles and the unpolygons in between. There are 2^k triangle regions and the remaining are unpolygons. The 2^k terms in this form cancel (similar to (37)) to leave (38).

$$\mathit{Untrigons}_k = \mathit{ThreeB}_k + \tfrac{1}{2}\,\mathit{ThreeNonB}_k - V_k - \mathit{QI}_k - 2^k - 1$$

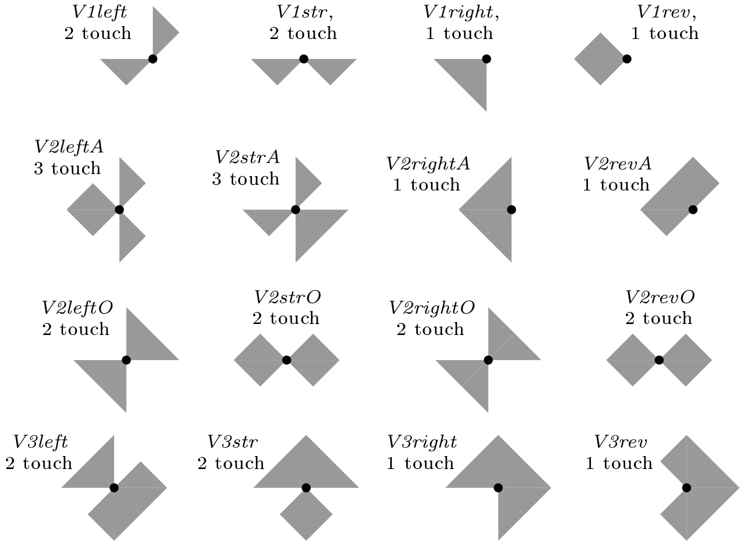
Untrigons has the same recurrence as *Trigons*. Difference from (36),(38) is

$$\begin{aligned} \mathit{Trigons}_k - \mathit{Untrigons}_k &= \tfrac{1}{2}\big(\mathit{V1sub3}_{k+1} - 1\big) \\ &= 1, 2, 4, 7, 13, 24, 46, 85, 159, 296, 554, 1025, \ldots \end{aligned}$$

This difference grows as r_{18} as from (29), which is slower than r of *poly9*. So ratio *Untrigons* to *Trigons* approaches 1, and from below since $\mathit{V1sub3} \geq 2$.

$$\frac{\mathit{Untrigons}_k}{\mathit{Trigons}_k} \to 1 \qquad \text{as } k{\to}\infty, \text{ approached from below}$$

At a given curve vertex there can be 1, 2 or 3 polygon pieces touching. The 1,2,3 visits and their turn determine how many pieces touch. *V2leftA* etc are from page 52. *V1left* and *V3left* etc are likewise turns and 1 or 3 visits.



The number of touches are the same in *V1* and *V3* with respective turns, since the two are fill/unfill inverses.

Similar to (32), each turn type begins as left from the respective inward square and opens out successively.

$$\begin{aligned} \mathit{V1left}_k &= \mathit{QI1}_k + \mathit{V1left}_{k-4} \\ &= 0, 1, 2, 4, 6, 13, 24, 48, 84, 163, 304, 584, 1082, \ldots \\ \mathit{V3left}_k &= \mathit{QI3}_k + \mathit{V3left}_{k-4} \\ &= 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 6, 20, 56, 156, \ldots \end{aligned}$$

$$\begin{aligned} \mathit{V1str}_k &= \mathit{V1left}_{k-1} & \mathit{V3str}_k &= \mathit{V3left}_{k-1} \\ \mathit{V1right}_k &= \mathit{V1left}_{k-2} & \mathit{V3right}_k &= \mathit{V3left}_{k-2} \\ \mathit{V1rev}_k &= \mathit{V1left}_{k-3} & \mathit{V3rev}_k &= \mathit{V3left}_{k-3} \end{aligned}$$

The totals are, with +2 for start and end of the curve which have no turn so not included in the turn types,

$$\begin{aligned} \mathit{V1}_k &= \mathit{V1left}_k + \mathit{V1str}_k + \mathit{V1right}_k + \mathit{V1rev}_k + 2 \\ \mathit{V3}_k &= \mathit{V3left}_k + \mathit{V3str}_k + \mathit{V3right}_k + \mathit{V3rev}_k \end{aligned}$$

V2leftO etc opposing points all have the same touches so can be taken together. They are cumulative inward opposite squares,

$$\begin{aligned} \mathit{V2o}_k &= \mathit{QI2o}_k + \mathit{V2o}_{k-1} \\ &= 0, 0, 0, 0, 0, 0, 0, 0, 2, 6, 16, 40, 96, 204, \ldots \end{aligned}$$

Curve start and end are again not in these vertex with turn counts, but have just 1 polygon piece each. The number of points touched by 1, 2 or 3 polygon pieces are then

$$\begin{aligned}
\mathit{VTouch1}_k &= \mathit{V1right}_k + \mathit{V1rev}_k \ + \mathit{V2rightA}_k + \mathit{V2revA}_k \\
&\quad + \mathit{V3right}_k + \mathit{V3rev}_k \ + 2 \\
&= 2, 2, 2, 3, 5, 8, 13, 24, 46, 89, \ldots \\
\mathit{VTouch2}_k &= \mathit{V1left}_k + \mathit{V1str}_k \ + \mathit{V2o}_k + \mathit{V3left}_k + \mathit{V3str}_k \\
&= 0, 1, 3, 6, 10, 19, 37, 72, 134, 255, \ldots \\
\mathit{VTouch3}_k &= \mathit{V2leftA}_k + \mathit{V2strA}_k \\
&= 0, 0, 0, 0, 1, 3, 7, 15, 34, 72, \ldots
\end{aligned}$$

Polygon pieces also touch at the tops of each triangle. Two pieces touch in a *QI2o* type square, and all other inward squares are 1 touch.

$$\begin{aligned}
\mathit{QTouch1}_k &= \mathit{QI1}_k + \mathit{QI2a}_k + \mathit{QI3}_k \\
&= 1, 2, 4, 7, 14, 27, 54, 101, 198, 383, \ldots \\
\mathit{QTouch2}_k &= \mathit{QI2o}_k
\end{aligned}$$

So number of 1, 2 or 3 polygon piece touch locations are

$$\begin{aligned}
\mathit{Touch1}_k &= \mathit{VTouch1}_k + \mathit{QTouch1}_k \\
&= 3, 4, 6, 10, 19, 35, 67, 125, 244, 472, \ldots \\
\mathit{Touch2}_k &= \mathit{VTouch1}_k + \mathit{QTouch1}_k \\
&= 0, 1, 3, 6, 10, 19, 37, 74, 138, 265, \ldots \\
\mathit{Touch3}_k &= \mathit{VTouch3}_k
\end{aligned}$$

$$\begin{aligned}
&\mathit{PolyTouch3}(x) = (x^2{+}1)\,.\,\mathit{poly8alt}(x)\,.\,\mathit{poly9}(x) \\
&\mathit{PolyTouch1}(x) = \mathit{PolyTouch2}(x) \ = (x{-}1)\,.\,\mathit{PolyTouch3}(x)
\end{aligned}$$

There is one *ThreeB* side anti-clockwise around at each polygon point so total weighted by touches is

$$\mathit{ThreeB}_k = \mathit{Touch1}_k + 2\,\mathit{Touch2}_k + 3\,\mathit{Touch3}_k$$

or unweighted is points plus inward squares, which together are next level points

$$\mathit{V1}_{k+1} + \mathit{V2}_{k+1} + \mathit{V3}_{k+1} = \mathit{Touch1}_k + \mathit{Touch2}_k + \mathit{Touch3}_k$$

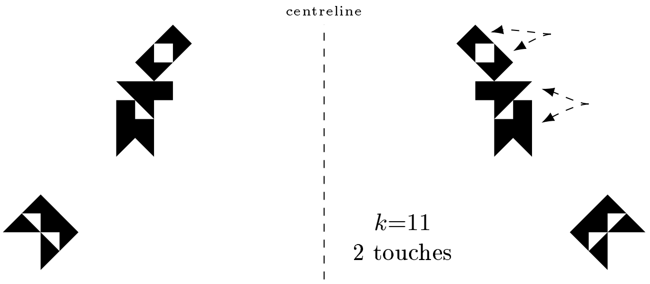
Each polygon piece variously touches other pieces, and possibly touches itself. A piece has a touch at each *Touch2* or *Touch3* location. Each of those are on 2 or 3 polygon pieces respectively. So the mean number of touches at a piece is, using the various *poly9* parts,

$$\begin{aligned}
\frac{2\,\mathit{Touch2}_k + 3\,\mathit{Touch3}_k}{\mathit{Trigons}_k} &\to \frac{1}{488504}\left(\begin{array}{c} 371728 + 1109244r \ - 1142344r^2 \\ + 1003322r^3 \ - 677148r^4 \ + 270311r^5 \\ - 67483r^6 \ - 68763r^7 \ + 44657r^8 \end{array}\right) \\
&= 3.589245\ldots
\end{aligned}$$

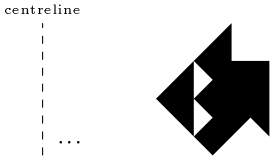
Or mean proportion of a polygon piece's points which touch 1 or more other pieces, including self-touches,

$$\frac{2\,\mathit{Touch2}_k + 3\,\mathit{Touch3}_k}{\mathit{ThreeB}_k} \to \frac{1}{1052288}\left(\begin{array}{c} 675976\ +367080r\ -664352r^2 \\ +431384r^3\ -176436r^4\ -3018r^5 \\ +164727r^6\ -182548r^7\ +56145r^8 \end{array}\right)$$
$$= 0.632922\ldots$$

It's possible for a given polygon piece to touch the same other piece at multiple places. The means above count all such. The first pieces with 2 same touches occur in k=11.

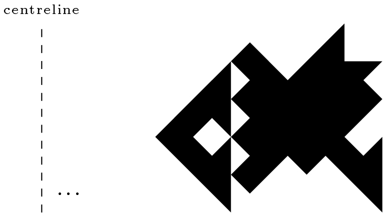


The lower touches are a piece with 2 touches each to 2 others. Pieces can touch more than 2 times too. The first 3 same touches are in k=14,



k=14 3 touch,
with same in
mirror image
across
centreline

The first 4 same touches are in k=16. This is the above k=14 pieces location expanded twice. The left piece here is the first with a self-touch. The indentation into it is the inverse of a *Tee* from (28).

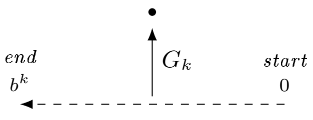


k=16,
with same in
mirror image
across
centreline

9 Centroid

Theorem 17. *Consider the C curve level k to have each segment length 1 and mass uniformly distributed along its length.*

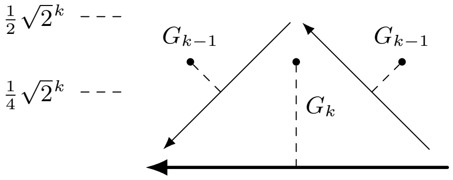
The curve is symmetric in horizontal mirror image so the centroid of its segments is on a line midway between and perpendicular to the endpoints.



$$G_k = \tfrac{1}{2}\left(\sqrt{2}^k - \frac{1}{\sqrt{2}^k}\right)$$
$$= 0,\ \tfrac{\sqrt{2}}{4},\ \tfrac{3}{4},\ \tfrac{7\sqrt{2}}{8},\ \tfrac{15}{8},\ \tfrac{31\sqrt{2}}{16},\ \tfrac{63}{16},\ \ldots$$

Proof. k=0 is a single line segment and the centroid is its midpoint, so $G_0 = 0$.

The centroid of curve level k is the midpoint of the centroids of its two $k{-}1$ sub-curves, suitably rotated and shifted.



$$G_k \;= \tfrac{1}{4}\sqrt{2}^k + \tfrac{1}{\sqrt{2}}G_{k-1} \;= \tfrac{1}{4}\sum_{j=0}^{k-1}\sqrt{2}^{k-j}\big(\tfrac{1}{\sqrt{2}}\big)^j \qquad \text{with } G_0 = 0 \qquad \square$$

Second Proof of Theorem 17. The centroid can also be calculated from the segments in each direction $S(k,d)$ of theorem 4.

$$G_k = \sqrt{2}\left(G_{k-1} + \tfrac{1}{2^{k-1}}\Big(\tfrac{\overline{b}}{\sqrt{2}}\Big)^{k-3}\sum_{d=0}^{3}\tfrac{1}{4}i^{d-1}S(k{-}1,d)\right) \qquad k \geq 1$$

Factor $\sqrt{2}$ for expansion maintains unit length of each segment. Each segment expands on the right, so its centroid moves by $\frac{1}{4}i^{d-1}$. Factor $1/2^{k-1}$ takes the mean of those moves.

$S(k,d)$ counts the first segment in fixed direction d=0, so a power of $\overline{b}/\sqrt{2}$ rotates to have the curve end vertical so the centroid is the real part. The imaginary parts cancel by symmetry. $\square$

$G_k \to \frac{1}{2}\sqrt{2}^k$ which is where the two sub-curves begin and end. Scaled to endpoints a unit length this is

$$\frac{G_k}{\sqrt{2}^k} \to \frac{1}{2}$$

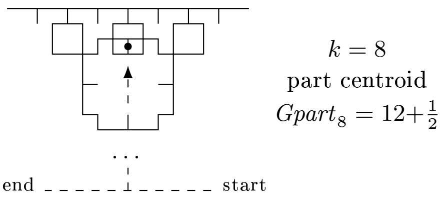
For an arbitrary unfolding angle θ a similar calculation gives

$$\begin{aligned} b_\theta &= 1 + e^{(\pi-\theta)i} \qquad |b_\theta| = \sqrt{2-2\cos\theta} \;= 2\sin\tfrac{\theta}{2} \\ G(k,\theta) &= \tfrac{1}{2}(\cos\tfrac{\theta}{2})\,|b_\theta|^{k-1} + (\sin\tfrac{\theta}{2})\,G_{k-1} \\ &= \tfrac{1}{2}(\cos\tfrac{\theta}{2})\sum_{j=0}^{k-1}(2\sin\tfrac{\theta}{2})^{k-1-j}(\sin\tfrac{\theta}{2})^j \qquad \text{with } G(0,\theta) = 0 \\ &= \frac{1}{2\tan\frac{\theta}{2}}(\sin\tfrac{\theta}{2})^k\,(2^k - 1) \qquad \theta > 0 \end{aligned}$$

and which scaled to a unit length between endpoints is

$$\frac{G(k,\theta)}{|b_\theta|^k} \to \frac{1}{2\tan\frac{\theta}{2}}$$

The centroid of a single curve part can be calculated similarly.



Theorem 18. *Consider C curve level k to have each segment length 1 and mass uniformly distributed along its length.*

The centroid of the segments comprising middle part k is distance $Gpart_k$ up from the line connecting curve start and end,

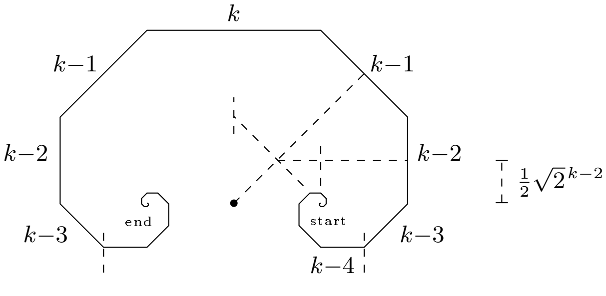
$$\begin{aligned} Gpart_k &= \frac{GpartTotal_k}{PartSegments_k} \\ &= \tfrac{11}{14}\sqrt{2}^k \; + \tfrac{1}{14} \frac{-[15, 27]\sqrt{2}^k \; + [-18,\, 27\sqrt{2}]}{2^k + [2, -2]} \\ &= 0, none,\ 1,\ \tfrac{5}{4}\sqrt{2},\ \tfrac{17}{6},\ \tfrac{59}{20}\sqrt{2},\ \tfrac{135}{22},\ \tfrac{173}{28}\sqrt{2}, \dots \end{aligned} \tag{39}$$

where GpartTotal is sum distances segment midpoints of the part

$$\begin{aligned} GpartTotal_k &= \tfrac{11}{42}(2\sqrt{2})^k \; + \tfrac{1}{6}[1, -7]\sqrt{2}^k \; + \tfrac{3}{14}[-2,\, 3\sqrt{2}] \\ &= 0, 0,\ 2,\ \tfrac{5}{2}\sqrt{2},\ 17,\ \tfrac{59}{2}\sqrt{2},\ 135,\ \tfrac{519}{2}\sqrt{2}, \dots \end{aligned}$$

Part k=1 is empty so it has no centroid, $Gpart_1$ = none.

Proof. The centroid of part k can be found from the whole curve by subtracting parts $k-1$ and below.



$GpartTotal_{k-1}$ is directed on a line perpendicular to its $k-1$ sub-curve start and end, so an offset $\frac{1}{2}\sqrt{2}^{k-2}$ upwards and this is to be multiplied by its $PartSegments_{k-1}$.

$GpartTotal_{k-2}$ has the same offset up as $k-1$ does and is to be multiplied by its $PartSegments_{k-2}$. It is directed horizontally so no centroid term, just this position offset.

$GpartTotal_{k-3}$ is similar to $k-1$ but directed downward from its $k-3$ sub-curve middle of the start to end, of which up from the k line at offset $\frac{1}{2}\sqrt{2}^{k-4}$.

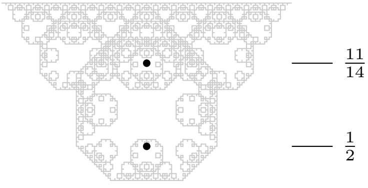
The remaining parts $k-4$ down can be joined together to make a $k-4$ curve pointing downwards, with an extra copy of middle part $k-4$. The net result, with $Gtotal_k = 2^k G_k$ for the whole curve total segment midpoints,

$$\begin{aligned}
\mathit{GpartTotal}_k = {} & \mathit{Gtotal}_k \\
& - 2\Big(\mathit{PartSegments}_{k-1}\,\tfrac{1}{2}\sqrt{2}^{k-2} + \mathit{GpartTotal}_{k-1}\,\tfrac{1}{2}\sqrt{2}\Big) \\
& - 2\Big(\mathit{PartSegments}_{k-2}\,\tfrac{1}{2}\sqrt{2}^{k-2}\Big) \\
& - 2\Big(\mathit{PartSegments}_{k-3}\,\tfrac{1}{2}\sqrt{2}^{k-4} + \mathit{GpartTotal}_{k-3}\,\tfrac{-1}{2}\sqrt{2}\Big) \\
& - \big(-\mathit{Gtotal}_{k-4} - \mathit{GpartTotal}_{k-4}\big)
\end{aligned}$$

This is an order 4 recurrence and simplifies to powers of $2\sqrt{2}$ and $\sqrt{2}$ with periodic terms. ☐

Scaled to curve endpoints a unit length the limit is coefficient $\frac{11}{14}$ in *Gpart* at (39). This is above the corresponding limit $\frac{1}{2}$ for the whole curve G_k.

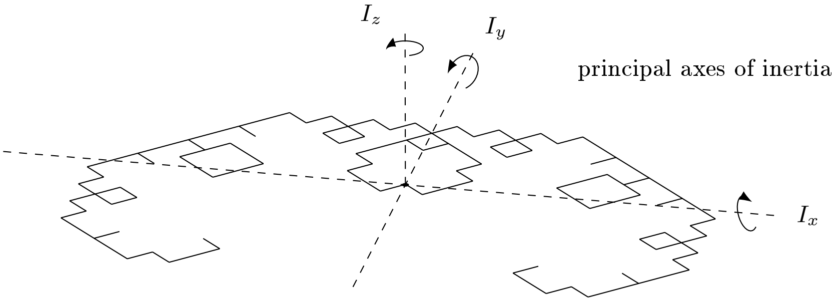
$$\frac{\mathit{Gpart}_k}{\sqrt{2}^k} \to \frac{11}{14}$$



10 Moment of Inertia

The mass moment of inertia $I = \sum mr^2$ of a rigid body rotating around a given axis is the ratio of torque to angular acceleration, similar to the way mass is the ratio of force to linear acceleration.

Any plane figure which is symmetric in mirror image has principal axes aligned to that symmetry. This is since the product of inertia $\sum mxy = 0$ if all points occur as pairs x, y and $-x, y$. For the C curve this means axes parallel and perpendicular to the endpoints, through the centre of gravity.



Theorem 19. *Consider the C curve with unit length segments and a point mass 1 at the midpoint of each segment, for total mass 2^k. The moment of inertia tensor about the centre of gravity is*

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

$$I_x(k) = \tfrac{1}{8}\Big(4^k - (2k{-}1)\,2^k - 2\Big)$$

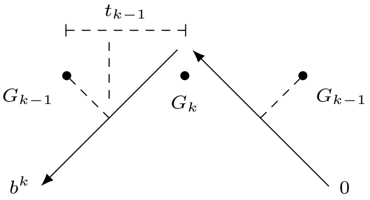
$$= 0,\ 0,\ \tfrac{1}{4},\ \tfrac{11}{4},\ \tfrac{71}{4},\ \tfrac{367}{4},\ \tfrac{1695}{4},\ \tfrac{7359}{4},\dots \qquad \tfrac{1}{8}\,\text{A286778}$$

$$\begin{aligned} I_y(k) &= I_x(k{+}1) - I_x(k) \\ &= \tfrac{1}{8}\Big(3.4^k - (2k{+}3)\,2^k\ \Big) \\ &= 0,\ \tfrac{1}{4},\ \tfrac{5}{2},\ 15,\ 74,\ 332,\ 1416,\ 5872,\dots \qquad 2^{k-3}{\times}\text{A050488} \end{aligned}$$

$$I_z(k) = I_x(k) + I_y(k) \quad = I_x(k{+}1)$$

Proof. For $k{=}0$ there is a single point mass and it has inertia 0 for all axes.

For $k \ge 1$ the curve is of 2 copies of level $k{-}1$ at right angles. Let t_{k-1} be the distance from the G_{k-1} centroid to the G_k centroid.



$I_z(k)$ from two copies of $I_z(k{-}1)$

$$t_{k-1} = \tfrac{1}{\sqrt{2}}\,G_{k-1} + \tfrac{1}{4}\sqrt{2}^k \quad = \frac{2^k{-}1}{2.\sqrt{2}^k}$$

The distances to the axes are scaled by $1/\sqrt{2}$ for the 45° rotations, which squared and with two sub-parts is factor $2\big(\frac{1}{\sqrt{2}}\big)^2 = 1$.

$$I_x(k) = I_x(k{-}1)\ +\ I_y(k{-}1) \qquad = I_z(k{-}1) \qquad k \ge 1 \tag{40}$$

$$I_y(k) = I_x(k{-}1)\ +\ I_y(k{-}1)\ +\ 2.2^{k-1}.t_{k-1}^2 \tag{41}$$

$I_z = I_x + I_y$ is true of any plane figure, since squared distances to the axes are $I_z = \sum m(x^2{+}y^2)$, $I_x = \sum m(y^2{+}z^2)$, $I_y = \sum m(x^2{+}z^2)$ and for a plane figure $z = 0$. So (40) is $I_x(k) = I_z(k{-}1)$. Add (40),(41) for I_z recurrence, which is also the parallel axis theorem,

$$\begin{aligned} I_z(k) &= 2I_z(k{-}1)\ +\ 4^{k-1} - 2^{k-1} + \tfrac{1}{4} \\ &= \sum_{j=0}^{k-1} 2^j\left(4^{k-1-j} - 2^{k-1-j} + \tfrac{1}{4}\right) \qquad \text{with } I_z(0) = 0 \\ &= \frac{4^k - 2^k}{4-2}\ -\ k\,2^{k-1}\ +\ \tfrac{1}{4}(2^k - 1) \\ &= \tfrac{1}{4}\Big(2.4^k - (2k{+}1)2^k - 1\Big) \end{aligned}$$

and from which I_y as first differences per (40). □

For an arbitrary unfolding angle θ the calculation becomes, using $G(k{-}1,\theta)$ and b_θ from section 9,

$$t(k{-}1,\theta)\ = G(k{-}1,\theta)\,\cos\tfrac{\theta}{2} + \tfrac{1}{4}\,|b_\theta|^k$$

$$\begin{aligned}
&= \tfrac{1}{2}\,(2^{k-1} - \cos^2\tfrac{\theta}{2})\,\sin^{k-2}\tfrac{\theta}{2} \\
I_z(k,\theta) &= 2\,I_z(k{-}1,\theta) + 2.2^{k-1}\,t(k{-}1,\theta)^2 \qquad \text{parallel axis theorem} \\
&= 2^k \sum_{j=0}^{k-1} t(k{-}j{-}1,\theta)^2 \\
&= \begin{cases}
I_z(k) & \text{if } \theta = \frac{\pi}{2} \\
(k - \frac{9}{4})\,2^k + 3 - \frac{3}{4}(\frac{1}{2})^k & \text{if } \theta = \frac{\pi}{3} \\
\frac{2}{1-2\cos\theta}\,(8\sin^2\frac{\theta}{2})^{k-1} + \frac{1+\cos\theta}{\cos\theta}\,(4\sin^2\frac{\theta}{2})^{k-1} & \\
\quad - \frac{1}{4}\,(1+\cos\theta)\,(2\sin^2\frac{\theta}{2})^{k-1} - \frac{1}{4}\big(1+\frac{2}{(1-2\cos\theta)\cos\theta}\big)\,2^k &
\end{cases}
\end{aligned}$$

The powers form for the general case is written with exponents $k{-}1$ since it slightly simplifies the coefficients.

The special cases for $\theta = \frac{\pi}{2}, \frac{\pi}{3}$ are where the denominators $\cos\theta$, $1{-}2\cos\theta$ respectively in the powers form would be zero. There is no discontinuity there, just sum of powers $\sum_{j=0}^{k-1} a^j = a^k/(a{-}1)$ if $a{\neq}1$ but $=k$ if $a{=}1$.

Case $\theta{=}\pi$ is full unfolding to a straight line and inertia is a simple sum over the squared distances to the segment midpoints for $I_z(k,\pi) = \frac{1}{12}2^k(4^k - 1)$. This is the first and last terms of the general case, the middle two factors have $1{+}\cos\pi = 0$.

$\sin^2\frac{\theta}{2}$ in the general form reduces the powers 8^k etc. For $\theta{=}\pi$ the top term $8\sin^2\frac{\theta}{2}$ is full 8^k. For right-angle $\theta{=}\frac{\pi}{2}$ have $\sin^2\frac{\theta}{2} = \frac{1}{2}$ so the top term reduces to 4^k as per $I_z(k)$ (that part of the sum is fine, it is the middle part where $4\sin^2\frac{\theta}{2} = 2$ which is the special case). For $\theta{=}\frac{\pi}{3}$ have $2\sin^2\frac{\theta}{2} = \frac{1}{2}$ which is where the $(\frac{1}{2})^k$ arises in its special case.

References

[1] Antoine-Augustin Cournot, “Solution d’un Problème d’Analyse Combinatoire”, Bulletin des Sciences Mathématiques, Physiques et Chimiques, item 34, volume 11, 1829, pages 93–97.

`http://books.google.com.au/books?id=B-v-eXuvoG4C`

[2] P. Duvall and J. Keesling, “The Dimension of the Boundary of the Lévy Dragon”, International Journal of Mathematics and Mathematical Sciences, volume 20, number 4, 1997, pages 627–632.

Preprint “The Hausdorff Dimension of the Boundary of the Lévy Dragon”

`http://at.yorku.ca/p/a/a/h/08.htm`

[3] Paul Lévy, “Les Courbes Planes ou Gauches et les Surfaces Composée de Parties Semblables au Tout”, (Plane or Space Curves and Surfaces Consisting of Parts Similar to the Whole), Journal de l’École Polytechnique, number 7, July 1938 pages 227–247 and number 8, October 1938, pages 249–292.

`http://gallica.bnf.fr/ark:/12148/bpt6k57344323/f53.image`
`http://gallica.bnf.fr/ark:/12148/bpt6k57344820`

[4] Christian Ramus, “Solution Générale d’un Problème d’Analyse Combinatoire”, Journal für die Reine und Angewandte Mathematik (Crelle’s journal), volume 11, 1834, pages 353–355.

`http://gdz.sub.uni-goettingen.de/en/dms/load/toc/?PPN=PPN243919689_0011`

[5] Robert S. Strichartz and Yang Wang, “Geometry of Self-Affine Tiles I”, Indiana University Mathematics Journal, volume 48, number 1, 1998, pages 1–23.
`http://www.mth.msu.edu/~ywang/reprints/boundaryi.pdf`

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