http://user42.tuxfamily.org/terdragon/index.html

Iterations of the Terdragon Curve

Kevin Ryde

September 2018, Draft 12

Abstract

Various results on the terdragon curve, including coordinates, area, boundary, enclosure sequence, convex hull, centroid, moment of inertia, some trees and fractionals.

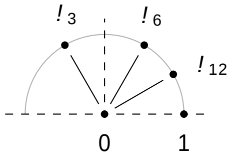
Contents

Notation

Various coordinates and other expressions use complex 3rd, 6th and 12th roots of unity, usually to express directions.

$$\omega_3 = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i \;=\; e^{2\pi i/3} \qquad \text{3rd root of unity, } 120^\circ$$

$$\begin{aligned} \omega_6 &= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i &&= e^{2\pi i/6} = \omega_3 + 1 && \text{6th root of unity, } 60^\circ \\ \omega_{12} &= \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i &&= e^{2\pi i/12} && \text{12th root of unity, } 30^\circ \end{aligned}$$



A few formulas have terms going in a repeating pattern of say 4 values according as an index $k \equiv 0\,\text{to}\,3 \bmod 4$. It's convenient to write them as for example

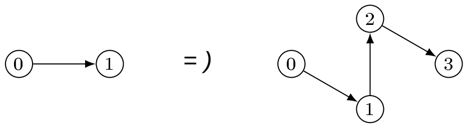
$$[5,\ 8,\ -5,\ 9] \qquad \text{values according as } k \bmod 4$$

meaning 5 when $k \equiv 0 \bmod 4$, or 8 when $k \equiv 1 \bmod 4$, etc. Likewise periodic patterns of other lengths, usually at most 8.

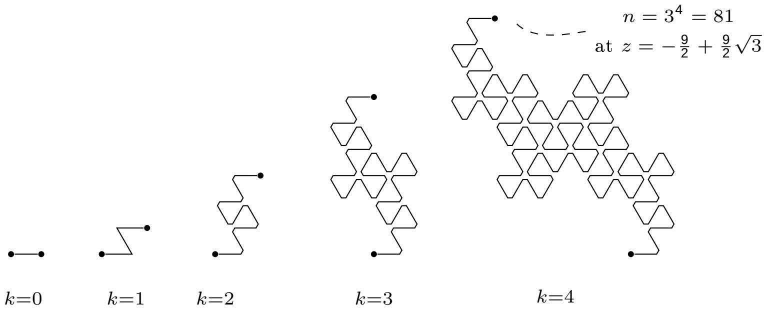
Periodic patterns like this can also be expressed using powers of -1 or i or other roots of unity, but except in simple cases that tends to be less clear than the values.

1 Terdragon Curve

The terdragon curve by Davis and Knuth[3] is defined recursively as a repeated replacement of each line segment by 3 segments in an "S" shape



The curve touches at vertices. The following diagram has the vertices chamfered off to better see the turns and joins.



Davis and Knuth show the terdragon is non-crossing and plane filling from the revolving cubic representations of its vertices. This can also be seen geometrically.

Theorem 1 (Davis and Knuth).

Consider an infinite triangular grid with unit line segments connecting the points. Each line segment expands to the base pattern as follows. The corners of the new line segments are chamfered off here to show how they meet the expansions from other lines but do not cross.

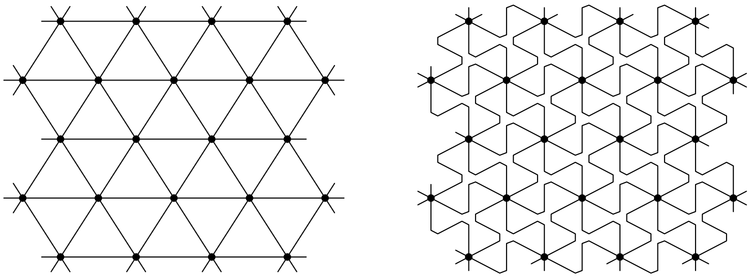
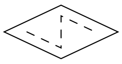


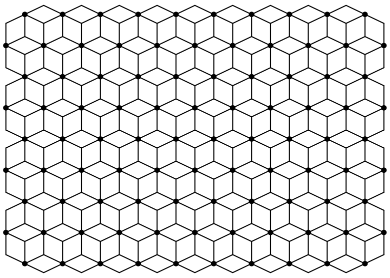
Figure 1: segment expansions

The expanded segments are the same grid pattern rotated by 30°.

Any subset of the full grid expands to a new bigger set with the number of crossings unchanged. The terdragon curve begins with a single line segment which is such a subset with no crossings and so on repeated expansions has no crossings. □

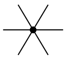
The expansion replaces each line segment with a rhombus shaped three segments. This is a classical tiling pattern[9].

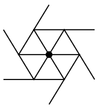




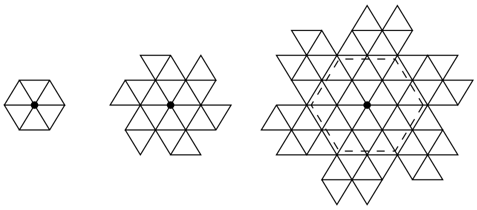
Theorem 2 (Davis and Knuth).
60°

The initial 6 line segments expand





Take the central 2×2 hexagon. With two expansions it grows



The dashed outline is a 4×4 hexagon at the origin. Each 2×2 hexagon (possibly overlapping) grows to at least 4×4. By repeated expansion they grow to an arbitrarily large hexagon at the origin. ☐

See end of subsection 10.1 for the actual diameter of 6 arm filling.

Number the points starting $n = 0$ at the origin. Per Davis and Knuth the replications give a turn sequence which is 120° turns according to n in ternary.

$$\mathit{turn}(n) = \begin{cases} +1 & \text{if } \mathit{LowestNonZero}(n) = 1 \\ -1 & \text{if } \mathit{LowestNonZero}(n) = 2 \end{cases} \qquad n \ge 1 \tag{1}$$

$$= -(-1)^{\mathit{LowestNonZero}(n)}$$

$$= + - ++ - + - ++ - ++ - + - + - ++ - + - ++ - ++ - + \dots$$

$$\mathit{turn}(3n) = \mathit{turn}(n), \quad \mathit{turn}(3n{+}1) = 1, \quad \mathit{turn}(3n{+}2) = -1 \tag{2}$$

$$\mathit{LowestNonZero}(n) = 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 1, 1, 2, \dots \quad n{\ge}1 \qquad \text{A060236}$$

Or next turn,

$$\mathit{turn}(n{+}1) = \begin{cases} +1 & \text{if } \mathit{LowestNonTwo}(n) = 0 \\ -1 & \text{if } \mathit{LowestNonTwo}(n) = 1 \end{cases} \qquad n \ge 0$$

$$= (-1)^{\mathit{LowestNonTwo}(n)}$$

$$\mathit{LowestNonTwo}(n) = 0, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, \dots \quad n{\ge}0 \qquad \text{A080846}$$

$3^k{-}1$ is entirely of 2-digits but is taken to have a 0 above the highest so $\mathit{LowestNonTwo}(3^k{-}1) = 0$.

$\mathit{turn}(n)$ and $\mathit{turn}(n{+}1)$ are related simply by $n{+}1$ changing low 2s into low 0s and increment the digit above.

n		*d*	2 2	ternary digits
n+1		*d*+1	0 0	

$$\mathit{LowestNonTwo}(n) \;=\; \mathit{LowestNonZero}(n{+}1) \;-\; 1$$

On a binary computer it can convenient to represent ternary digits in 2 bits each. Arndt[1] gives an example iterating *turn* like this with bits $00, 01, 10$ to represent $0, 1, 2$ respectively and a search loop for carry propagation.

Another possibility is bits $00, 01, 11$. This allows a binary increment to propagate a carry through 2s. If it increments 01 to 10 then a normalize up to

11 is necessary. Representing ternary 1 by bits 01 (rather than 10) is convenient since the lowest non-0 digit is then determined by bit above lowest 1-bit and that can be found by bit-twiddling.

nbits has bits $00, 01, 11$ representing ternary digits $0, 1, 2$ A023713

$$\mathit{turn}(\mathit{nbits}) = \begin{cases} +1 & \text{if } \mathit{BitAboveLowestOne}(\mathit{nbits}) = 0 \\ -1 & \text{if } \mathit{BitAboveLowestOne}(\mathit{nbits}) = 1 \end{cases}$$

$$\mathit{increment}(\mathit{nbits}) = \mathit{PostIncFix}(\mathit{nbits} + 1)$$

$$\mathit{PostIncFix}(n) = \mathrm{BITOR}(n,\ \mathrm{BITAND}(1010\ldots101_2,\ \mathrm{RIGHTSHIFT}(n)))$$

$$\mathit{BitAboveLowestOne}(n) = \begin{cases} 0 & \text{if } \mathrm{BITAND}(n, \mathit{MaskAboveLowestOne}(n)) = 0 \\ 1 & \text{if } \mathrm{BITAND}(n, \mathit{MaskAboveLowestOne}(n)) \neq 0 \end{cases}$$

$$= 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, \ldots \qquad \text{A038189}$$

$$\mathit{MaskAboveLowestOne}(n) = \mathrm{BITXOR}(n,\ n{-}1) + 1 \qquad n \geq 1$$

$$= 2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 32, \ldots \qquad \text{A171977}$$

Predicates for left and right turns are

$$\mathit{TurnLpred}(n) = \begin{cases} 1 & \text{if } n{\geq}1 \text{ and } \mathit{LowestNonZero}(n) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= 1{,}0{,}1{,}1{,}0{,}0{,}1{,}0{,}1{,}1{,}0{,}1{,}1{,}0{,}0{,}1{,}0{,}\ldots \quad n{\geq}1 \qquad \text{A137893,A189673}$$

$$\mathit{TurnRpred}(n) = \begin{cases} 1 & \text{if } n{\geq}1 \text{ and } \mathit{LowestNonZero}(n) = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= 0{,}1{,}0{,}0{,}1{,}1{,}0{,}1{,}0{,}0{,}1{,}0{,}0{,}1{,}1{,}0{,}1{,}\ldots \quad n{\geq}1 \qquad \text{A080846,A189640}$$

Generating functions for these sequences follow by considering the ternary digits of those n which are a left or right turn. A left turn is k low zeros then digit 1 so $n = 3^k + m.3^{k+1}$ for integer m. Generating function $1/(1{-}x^{3^{k+1}})$ is 1 at $m.3^{k+1}$ then multiply x^{3^k} to add 3^k. Similarly a right turn is k low zeros then digit 2 so $n = 2.3^k + m.3^{k+1}$ which is multiply by $x^{2.3^k}$ to add 2.3^k,

$$\mathit{gTurnLpred}(x) = \sum_{k=0} \frac{x^{3^k}}{1 - x^{3^{k+1}}} \qquad \mathit{gTurnRpred}(x) = \sum_{k=0} \frac{x^{2.3^k}}{1 - x^{3^{k+1}}} \tag{3}$$

With $\mathit{turn}(n) = \mathit{TurnLpred}(n) - \mathit{TurnRpred}(n)$ a generating function for turn is the difference. Factor $1{-}x^{3^k}$ cancels from numerator and denominator

$$\mathit{gturn}(x) = \sum_{k=0} \frac{x^{3^k} - x^{2.3^k}}{1 - x^{3^{k+1}}} = \sum_{k=0} \frac{x^{3^k}}{1 + x^{3^k} + x^{2.3^k}} \tag{4}$$

Paul D. Hanna in OEIS A080846 gives a generating function for $\mathit{TurnRpred}$ based on a generating function for total turn (dir ahead in subsection 1.3). Shifted to the numbering here first turn at $n{=}1$ term x^1 this is

$$\mathit{gTurnRpred}(x) = \tfrac{1}{2}\frac{x}{1-x} - \tfrac{1}{2}\sum_{k\geq 0} \frac{x^{3^k}}{1+x^{3^k}+x^{2.3^k}}$$

This can be thought of as changing turn form (4) values from ±1 to 0,1 by $\mathit{TurnRpred}(n) = \tfrac{1}{2}\left(1 - \mathit{turn}(n)\right)$.

If a generating function for just an initial part of the sequence is required then stopping the sum (either form) at k suffices for $n < 3^{k+1}$ where the next term would begin (a left turn at k+1 low zeros and digit 1 above).

On expanding the curve, 2 turns are inserted at each segment. A segment is before, after, and between each existing turn.

$$\begin{array}{ccccccccccl} & \boxed{\text{L}} & & \boxed{\text{R}} & & \boxed{\text{L}} & & \boxed{\text{L}} & & & \text{existing turns} \\ \text{LR} & & \text{LR} & & \text{LR} & & \text{LR} & & \text{LR} & & \text{new turns} \end{array} \qquad \text{Figure 2}$$

The R and L added each side of an existing turn become runs either RR or LL according as that existing turn is R or L. So the run lengths in the turn sequence are an initial 1 then pairs either 1,2 or 2,1 according as $\mathit{turn} = +1$ or -1 respectively. Counting the first run as m=0,

$$\begin{aligned} \mathit{TurnRun}(m) &= \begin{cases} 1 & \text{if } m{=}0 \ \text{(lefts)} \\ \tfrac{3}{2} + \tfrac{1}{2}\mathit{turn}(\tfrac{m}{2}) & \text{if } m \text{ even } \geq 2 \ \text{(lefts)} \\ \tfrac{3}{2} - \tfrac{1}{2}\mathit{turn}(\tfrac{m+1}{2}) & \text{if } m \text{ odd} \ \text{(rights)} \end{cases} \\ &= \begin{cases} 1 & \text{if } m{=}0 \\ \tfrac{3}{2} + \tfrac{1}{2}(-1)^m \mathit{turn}\left(\left\lceil \tfrac{m}{2} \right\rceil\right) & \text{if } m \geq 1 \end{cases} \\ &= 1,\ 1,2,\ 2,1,\ 1,2,\ 1,2,\ 2,1,\ 2,1,\ 1,2,\ldots \\ \mathit{turn} &= \quad\ \ +1,\ \ -1,\ \ +1,\ \ +1,\ \ -1,\ \ -1,\ \ +1, \end{aligned}$$

$$\mathit{gTurnRun}(x) = -\tfrac{1}{2} + \tfrac{3}{2}\frac{1}{1-x} + \tfrac{1}{2}\left(1 - \frac{1}{x}\right)\mathit{gturn}(x^2)$$

For a curve of finite k the run lengths end with a final 1 which is like the initial 1. By symmetry the run length sequence for finite k is equal to its reversal.

The n which is the start of a run follows from figure 2 turns too. In each LR the left $n \equiv 1 \bmod 3$ is the start of a run unless preceded by an existing turn L. The right at $n \equiv 2 \bmod 3$ is always the start of a run. Expressing this with an index $m \geq 0$,

$$\begin{aligned} \mathit{TurnRunStart}(m) &= 1 + \sum_{j=0}^{m-1} \mathit{TurnRun}(j) \\ &= \tfrac{3}{2}m + \begin{cases} 1 - \mathit{TurnLpred}(\tfrac{3}{2}m) & \text{if } m \text{ even} \\ \tfrac{1}{2} & \text{if } m \text{ odd} \end{cases} \\ &= \tfrac{3}{2}m + \begin{cases} 1 & \text{if } m{=}0 \\ \mathit{TurnLpred}(m) & \text{if } m \text{ even} \geq 2 \end{cases} \qquad (5) \\ &= 1, 2, 3, 5, 7, 8, 9, 11, 12, 14, 16, 17, 19, \ldots \end{aligned}$$

Form (5) holds since *TurnLpred* parameter can omit factor 3 using $\mathit{turn}(3n) = \mathit{turn}(n)$ which is just an extra low 0 digit. And multiply 2 removes factor $\tfrac{1}{2}$

using $\mathit{turn}(2n) = -\mathit{turn}(n)$ since factor 2 flips the lowest non-zero 1↔2. This negate swaps to *TurnRpred* and the $1-$ is back to *TurnLpred*, for $m{\neq}0$.

Theorem 3. m n
$m{=}0$

$$\mathit{TurnLeft}(m) = \begin{cases} 1 & m{=}0 \\ 3^k + \mathit{TurnLeft}\ \ m - \tfrac{1}{2}(3^k{+}1) & m < 3^k \\ 2.3^k + \mathit{TurnLeft}\ \ m - 3^k & m \geq 3^k \end{cases} \tag{6}$$

$$k \qquad \tfrac{1}{2}(3^k+1) \leq m$$

$$= 1, 3, 4, 7, 9, 10, 12, 13, 16, 19, 21, 22, \dots \qquad \text{A026225}$$

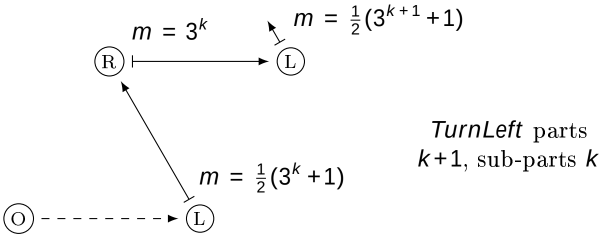
$$\mathit{TurnRight}(m) = \begin{cases} 3^k\ + \mathit{TurnRight}\ \ m - \tfrac{1}{2}(3^k{-}1) & m < 3^k{-}1 \\ 2.3^k & m = 3^k{-}1 \\ 2.3^k\ + \mathit{TurnRight}\ \ m - 3^k & m \geq 3^k \end{cases} \tag{7}$$

$$k \qquad \tfrac{1}{2}(3^k-1) \leq m$$

$$= 2, 5, 6, 8, 11, 14, 15, 17, 18, 20, 23, 24, \dots \qquad \text{A026179}$$

In an expansion level k there are 3^k segments and $3^k - 1$ turns between them. Since the curve is symmetric in 180° rotation there are half lefts and half rights $\frac{1}{2}(3^k - 1)$.

The recurrences follow from the curve sub-parts. Expansion level $k{+}1$ comprises sub-parts level k,



The m which is the first L in part 1 is the number of L preceding. There are $\frac{1}{2}(3^k{-}1)$ in part 0, plus the L between parts 0 and 1. Taking k as the biggest with $\frac{1}{2}(3^k{+}1) \leq m$ has m in either part 1 or 2.

The m which is the first L of part 2 is the number of L preceding there, which are a further $\frac{1}{2}(3^k{-}1)$ in part 1 for total 3^k. Comparing m to 3^k thus determines whether it is in part 1 or 2. Subtracting the respective start gives an m within level k, and add $n = 3^k$ or $n = 2.3^k$ as the starting points.

Likewise *TurnRight*, but its start of part 1 is without $+1$ for the L so $\frac{1}{2}(3^k{-}1)$. The R between parts 1 and 2 is $m{=}3^k{-}1$ and is an exception in the cases since the end of the k part would have an L there. ☐

Both *TurnLeft* and *TurnRight* are close to $2m$, roughly since the number of each is the same at the end of an expansion level. Or algebraically in (6), (7) a $\frac{1}{2}(3^k{\pm}1)$ subtracted from m is 3^k added to n, and in part 2 similarly 3^k subtracted from m is 2.3^k added to n. Offsets from $2m$ can be expressed

$$\begin{aligned}\mathit{TurnLeftO}(m) &= 2m - \mathit{TurnLeft}(m)\\ &= -1,-1,0,-1,-1,0,0,1,0,-1,-1,0,-1,-1,\ldots\\ \mathit{TurnRightO}(m) &= \mathit{TurnRight}(m) - 2m\\ &= 2,3,2,2,3,4,3,3,2,2,3,2,2,3,\ldots\end{aligned}$$

Substituting into (6),(7) gives recurrences

$$\mathit{TurnLeftO}(m) = \begin{cases} -1 & \text{if } m{=}0\\ \mathit{TurnLeftO}\left(m - \tfrac{1}{2}(3^k{+}1)\right) + 1 & \text{if } m < 3^k\\ \mathit{TurnLeftO}\left(m - 3^k\right) & \text{if } m \ge 3^k\end{cases}$$

$$\text{where } k \text{ biggest } \tfrac{1}{2}(3^k+1) \le m$$

$$\mathit{TurnRightO}(m) = \begin{cases} \mathit{TurnRightO}\left(m - \tfrac{1}{2}(3^k{-}1)\right) + 1 & \text{if } m < 3^k{-}1\\ 2 & \text{if } m = 3^k{-}1\\ \mathit{TurnRightO}\left(m - 3^k\right) & \text{if } m \ge 3^k\end{cases}$$

$$\text{where } k \text{ biggest } \tfrac{1}{2}(3^k-1) \le m$$

In part 2, the L and R turns between parts 0,1 and 1,2 balance, so offsets are unchanged on descending. In part 1 the preceding L is an extra, making smaller *TurnLeft*. The offsets thus grow according to how many middle parts,

$$\begin{aligned}\mathit{TurnLeftO}(m) &\ge -1\\ \mathit{TurnRightO}(m) &\ge 2\end{aligned}$$

The increments between successive turns L or R are

$$\begin{aligned}\mathit{dTurnLeft}(m) &= \mathit{TurnLeft}(m{+}1) - \mathit{TurnLeft}(m)\\ &= 2,1,3,2,1,2,1,3,3,2,1,3,2,1,2,1,3,\ldots && \text{A026141,A026171}\\ \mathit{dTurnRight}(m) &= \mathit{TurnRight}(m{+}1) - \mathit{TurnRight}(m)\\ &= 3,1,2,3,3,1,2,1,2,3,1,2,3,3,1,2,3,\ldots && \text{A026181,A131989}\end{aligned}$$

The expansions in figure 2 show these increments are always 1, 2 or 3. The m'th such increment can be expressed by recurrences.

$$\mathit{dTurnLeft}(m) = \begin{cases} 2,1 & \text{if } m = 0,1\\ \mathit{dTurnLeft}\left(m - \tfrac{1}{2}(3^k{+}1)\right) & \text{if } m < 3^k{-}1\\ 3 & \text{if } m = 3^k{-}1\\ \mathit{dTurnLeft}\left(m - 3^k\right) & \text{if } m \ge 3^k\end{cases}$$

$$\text{where } k \text{ biggest } \tfrac{1}{2}(3^k+1) \le m \text{ and } k \ge 1$$

$$\mathit{dTurnRight}(m) = \begin{cases} 3 & \text{if } m = 0\\ \mathit{dTurnRight}\left(m - \tfrac{1}{2}(3^k{-}1)\right) & \text{if } m < 3^k{-}2\\ 1 & \text{if } m = 3^k{-}2\\ 2 & \text{if } m = 3^k{-}1\\ \mathit{dTurnRight}\left(m - 3^k\right) & \text{if } m \ge 3^k\end{cases} \tag{8}$$

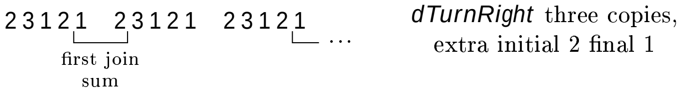
$$\text{where } k \text{ biggest } \tfrac{1}{2}(3^k-1) \le m \text{ and } k \ge 1$$

In these recurrences nothing is accumulated, just descend down m by parts until reaching one of the 1, 2 or 3 cases.

For *dTurnLeft*, case $m{=}3^k{-}1$ is the L of the last LR pair in part 1. It must step across the R between parts 1 and 2, so *dTurnLeft* $= 3$ there.

For *dTurnRight*, case $m{=}3^k{-}1$ is the R between parts 1 and 2, and $m{=}3^k{-}2$ preceding that is R of the last LR pair in part 1. The cases at (8) correspond to the recurrence given by Sloane in A131989 (indexes there starting from 1). That sequence is defined by a symbol substitution starting "$**|*$" and replace each $*$ by "$**|*$". This is the terdragon substitution where $*$ is a segment and $|$ is the R turn between parts 1 and 2. The sequence values are how many $*$ between successive $|$, and thus how far between successive R turns. The symbols as integers $1, 2$ are A133162.

Sloane again in A131989 gives a morphism replacement where copies of the sequence, and initial 2, are concatenated and the terms each side of the first join added together. That first join is a new left so sum distances each side to right turns.

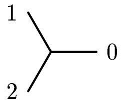


The total turn is a count of ternary 1 digits since each "1" sub-part is rotated +120° and sub-parts "0" and "2" are unchanged.

$$\operatorname{dir}(n) = \sum_{j=0}^{n-1} \textit{turn}(j) \quad = \text{ count ternary 1 digits in } n \tag{9}$$

$$= 0, 1, 0, 1, 2, 1, 0, 1, 0, 1, 2, 1, 2, 3, 2, 1, 2, 1, \ldots \qquad \text{A062756}$$

$\operatorname{dir}(n)$ mod 3 is a net direction East, North West or South West.



dir(n) mod 3
= 0; 1; 0; 1; 2; 1; 0; 1; 0; 1; 2; 1; 2; 0; 2; 1; : : :

The number of left and right turns from 1 to n inclusive are

$$\textit{TurnsL}(n) = \sum_{j=1} \textit{TurnLpred}(n)$$

$$= 1, 1, 2, 3, 3, 3, 4, 4, 5, 6, 6, 7, 8, 8, 8, 9, 9, 9, \ldots \qquad \text{A189674}$$

$$\textit{TurnsR}(n) = \sum_{j=1} \textit{TurnRpred}(n)$$

$$= 0, 1, 1, 1, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 7, 7, 8, 9, \ldots \qquad \text{A189641,A189672}$$

All turns are left or right so total lefts plus rights is simply n. The difference lefts minus rights is *dir*.

$$\textit{TurnsL}(n) + \textit{TurnsR}(n) = n \tag{10}$$

$$\mathit{TurnsL}(n) - \mathit{TurnsR}(n) = \mathit{dir}(n) \tag{11}$$

Sum and difference of (10),(11) are

$$\begin{aligned} \mathit{TurnsL}(n) &= \tfrac{1}{2}\ n + \mathit{dir}(n) \\ \mathit{TurnsR}(n) &= \tfrac{1}{2}\ n - \mathit{dir}(n) \end{aligned}$$

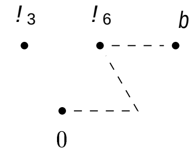
Clark Kimberling in OEIS A189674 and A189672 gives the following recurrences, with the first adapted here to *TurnsL* numbered first turn at n=1,

$$\begin{aligned} \mathit{TurnsL}(n) &= \mathit{TurnsL}\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor \\ \mathit{TurnsR}(n) &= \mathit{TurnsR}\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor \end{aligned}$$

These forms can be seen from the turn expansions in figure 2 (and general morphism expansions like A189674, A189672). $\mathit{TurnsL}(\lfloor n/3 \rfloor)$ counts turns in the "existing turns". Each is preceded by a new pair LR so $+\lfloor n/3 \rfloor$ further lefts. When $n \equiv 1, 2 \bmod 3$ the new L following the last "existing" is to be included too, so total $+\lfloor (n+2)/3 \rfloor$. Similarly *TurnsR*, but for it the following new R is only when $n \equiv 2 \bmod 3$, so $+\lfloor (n+1)/3 \rfloor$.

It's convenient to calculate terdragon curve coordinates in complex numbers using ω_3 or ω_6 roots of unity and a base b which is the end of a 3-segment unit expansion. The roots of unity act as rotations by 120° or 60°.

$$b = \omega_3 + 2 \ = \ \omega_6 + 1 \qquad \text{base}$$



Per Davis and Knuth, and counting vertices starting n=0 at the origin, point number n is given by ternary digits of $n = a_{k-1} \dots a_2 a_1 a_0$.

$$\mathit{digit}(a) = 0, 1, \omega_6 \quad \text{for } a = 0, 1, 2$$

$$\begin{aligned} \mathit{point}(n) = \ & b^{k-1}\, \mathit{digit}(a_{k-1}) && \text{high digit} \qquad (12) \\ & + b^{k-2}\, \mathit{digit}(a_{k-2})\ \omega_3^{\mathit{dir}(a_{k-1})} \\ & + b^{k-3}\, \mathit{digit}(a_{k-2})\ \omega_3^{\mathit{dir}(a_{k-1} a_{k-2})} \\ & \dots \\ & + b^{1} \quad \mathit{digit}(a_1) \quad \omega_3^{\mathit{dir}(a_{k-1} a_{k-2} \dots a_2)} \\ & + b^{0} \quad \mathit{digit}(a_0) \quad \omega_3^{\mathit{dir}(a_{k-1} a_{k-2} \dots a_2 a_1)} && \text{low digit} \\ = \ & 0,\ 1,\ \omega_6,\ 1{+}\omega_6,\ 2\omega_6,\ \omega_6,\ \omega_6,\ -1 + 2\omega_6,\ 2\omega_6,\ -1 + 3\omega_6,\ \dots \\ = \ & 0,\ 1,\ \tfrac{1}{2}{+}\tfrac{1}{2}\sqrt{3}i,\ \tfrac{3}{2}{+}\tfrac{1}{2}\sqrt{3}i,\ 1{+}\sqrt{3}i,\ \tfrac{1}{2}{+}\tfrac{1}{2}\sqrt{3}i,\ \sqrt{3}i,\ 1{+}\sqrt{3}i,\ \tfrac{1}{2}{+}\tfrac{3}{2}\sqrt{3}i,\ \dots \end{aligned}$$

Digits can be taken high to low as

$$\mathit{point}(3^k a_k + n_{k-1}) \;=\; b^k\,\mathit{digit}(a_k) \;+\; \mathit{point}(n_{k-1}).\omega_3^{\mathit{dir}(a_k)}$$

a_k is the highest digit and is located per the base pattern scaled by b^k. The n_{k-1} digits below it go in direction $\mathit{dir}(a_k)$ by multiplying ω_3. Repeated expansion is

$$\begin{aligned} \mathit{point}(n) &= b^k\,\mathit{digit}(a_k) \\ &\qquad + \omega_3^{\mathit{dir}(a_k)}\;\; b^{k-1}\,\mathit{digit}(a_{k-1}) \\ &\qquad\quad \dots \\ &\qquad\qquad + \omega_3^{\mathit{dir}(a_2)}\;\; b^{1}\,\mathit{digit}(a_1) \\ &\qquad\qquad\quad + \omega_3^{\mathit{dir}(a_1)}\;\; b^{0}\,\mathit{digit}(a_0) \end{aligned} \tag{13}$$

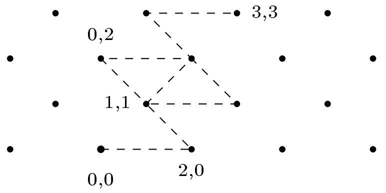
Digits can be taken low to high by segment replacement,

$$\mathit{point}(3n_1 + a_0) \;=\; \mathit{point}(n_1).b \;+\; \omega_3^{\mathit{dir}(n_1)}.\mathit{digit}(a_0) \tag{14}$$

a_0 is the low ternary digit and n_1 the digits above it. $\mathit{dir}(n_1)$ is the segment direction before expansion, so rotating the new base figure. This direction depends on all of n_1. Evaluating the nested (13) from innermost to outermost builds it successively by multiplying each direction onto all below.

For computer calculation, integer coordinates x, y representing $x + y\omega_3$ can be maintained. Or $x + y\omega_6$ if preferred. Multiplication by ω_3, ω_6 or b are then various integer additions or subtractions of x, y.

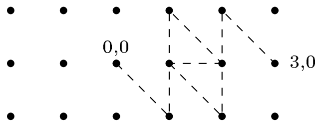
It's also possible to calculate with an x, y representing $\frac{1}{2}(x + y\sqrt{3}i)$ so that y is a purely imaginary term (vertical). In this case x, y are integers $x \equiv y \bmod 2$, ie. both even or both odd. The effect of plotting those x, y directly on an integer grid, without $\frac{1}{2}$ or $\sqrt{3}$ factors, is to flatten to right triangles height 1 base 2 (instead of equilateral triangles).



$x;y$ as $\frac{1}{2}(x + y\sqrt{3}i)$
for terdragon $k{=}2$ (15)

This form can be useful for a graphics display using every second pixel of a square grid. It avoids uneven spacing at small scales. If a factor $\sqrt{3}$ for equilateral triangles is used then it's necessary to round to an integer pixel and at resolutions near a few pixels this rounding becomes noticeable.

A grid of every second integer position is the same as a square grid rotated 45°. A further possible integer coordinate system is to take triangles on a 45° angle. This corresponds to integers x, y representing points $x\omega_6 + y\omega_3$.



$x;y$ as $x\omega_6 + y\omega_3$
for terdragon $k{=}2$

The low to high *point* formula (14) can be reversed to calculate n for a given segment. Suppose a segment is at $z = \textit{point}(n)$ in direction $d = 0, 1, 2 \equiv \textit{dir}(n)$ mod 3.

```
unpoint$(z, d)$            $d = 0, 1, 2$
  loop
    if $z=0$              then arm $= 2d$  end loop
    if $z=\omega_6$, $d\equiv 2$   then arm $= 1$   end loop
    if $z=-1$, $d\equiv 0$    then arm $= 3$   end loop
    if $z=\overline{\omega_6}$, $d\equiv 1$   then arm $= 5$   end loop
    $a = \begin{cases} 0 & \text{if } z \equiv 0 \mod b \\ 1 & \text{if } z \equiv 1 \mod b \\ 2 & \text{if } z \equiv \omega_6 \mod b \end{cases}$      ternary digit $a$
    $d \leftarrow d - \textit{dir}(a) \mod 3$
    $z \leftarrow (z - \textit{digit}(a).\omega_3^d) / b$
    $n$ digits low to high $\leftarrow a$
  end loop
  if arm even then $n$
  if arm odd  then $3^k - n$
  where $k$ is the number of digits of $n$ generated
```

z mod b determines the low ternary digit a of n since all terms of $\textit{point}(n)$ except the lowest are multiples of b, and in that low term $\omega_3 \equiv 1 \mod b$ so

$$z \equiv \textit{digit}(a_0) \mod b$$

The direction factor in (14) is all digits except a_0,

$$\textit{dir}(a_k \ldots a_1) = d - \textit{dir}(a_0)$$

Then the low digit is subtracted, b divided out, and the procedure repeated for the second lowest digit a_1, etc.

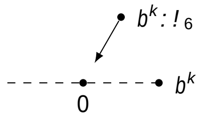
For segments in the terdragon curve starting in direction d=0 this ends with location z=0 and direction d=0.

For segments in a 120° rotated curve $z.\omega_3$, the procedure also ends with z=0 but direction d=1. This is since $\omega_3 \equiv 1 \mod b$ so that factor ω_3 does not change digits generated from z, and the initial d includes +1 for the rotation. Similarly segments in a 240° rotated curve $z.\omega_3^2$ reach z=0 and direction d=2.

For segments in a 60° rotated curve,

$$\textit{point}(n).\omega_6 = b^k.\omega_6 + \textit{point}(3^k - n).\omega_3^2$$

Geometrically this is starting at a 60° endpoint $b^k.\omega_6$ and going in direction d=2.



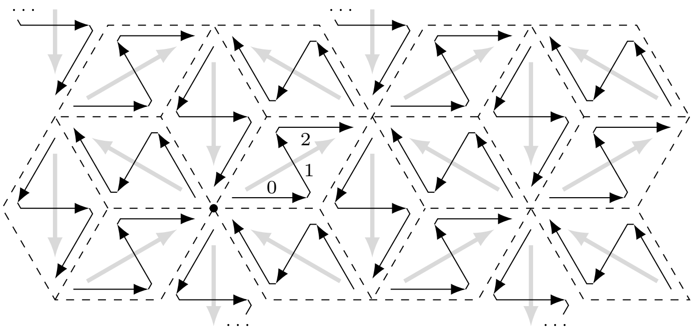
So the procedure gives digits of a 240° curve *point*$(3^k-n).\omega_3^2$, and loop ending $z=\omega_6$. Similarly for 180° and 300° rotated curves as arms 3 and 5. Notice these odd arms all take segment direction d as $0, 120, 240$, the same as the even arms. For the odd arms this is reverse along those arms, but the arm is not known until the end of the procedure.

If calculations are made in coordinates $x+y\omega_3$ then low digit a is simply

$$a = 0, 1, 2 \quad \equiv x{+}y \bmod 3$$

If using $x+y\omega_6$ then a similar $x-y \bmod 3$. Or every second point coordinates of (15) is $-x \bmod 3$

The geometric interpretation of the procedure is to find which rhombus shaped expansion from figure 1 contains the segment, then step back to the multiple of b which is its start. The rhombus tiling and directions are a repeating pattern and, depending on the x, y coordinate style used, can also be done in a 12×12 table lookup.



Each curve location z is visited 1, 2 or 3 times. Applying the *unpoint* procedure above for $d = 0, 1, 2$ gives the n which are those visits. For a given n, the other n_1, n_2 at the same location can be calculated from the ternary digits of n without the location as such.

Theorem 4. $n \geq 1$ n_1 n_2
n

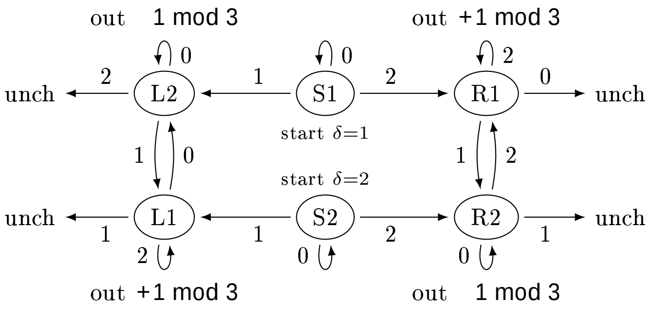


Figure 3:
other$(n; \;)$
by ternary
digits
low to high

$$\textit{other}\,(n, \delta) = \qquad \delta \qquad \pm 1 \bmod 3$$

$$\begin{array}{rlllllllllll}
\mathit{other}(n,1) = & 0, & 2, & 5, & 6, & 17, & 1, & 15, & 4, & 11, & 18,\ldots \\
= & 0, & -1, & 0, & -1, & -1, & 1, & 0, & 0, & 0, & -1,\ldots \\
\mathit{other}(n,2) = & 0, & 5, & 1, & 15, & 7, & 2, & 3, & 17, & 14, & 45,\ldots \\
= & 0, & -1, & 1, & -1, & 0, & 0, & 1, & -1, & 0, & -1,\ldots
\end{array}$$

$\delta = 1, 2$ $\mathit{dir}(n_\delta)$

$\equiv \mathit{dir}(n) + \delta \mod 3$ n $+1 \mod 3$

n $-1 \mod 3$

n

0 n

-1 $n_\delta = 3^k - \mathit{output}$

k

1 $n_\delta = 3^k - \mathit{output}$

Suppose m is the same location as n but direction $+\delta$, and a certain dz offset away from n.

$$\begin{aligned}
\mathit{dir}(m) &\equiv \mathit{dir}(n) + \delta \mod 3 \qquad (16)\\
\mathit{point}(m) &= \mathit{point}(n) + \omega_3^{\mathit{dir}(n)}.\mathit{dz}
\end{aligned}$$

Factor $\omega_3^{\mathit{dir}(n)}$ on dz makes dz relative to the direction of segment n, like a low term of point formula (12). This allows step (18) to require only the low digit of n.

The digits of m are to be determined from δ, dz and the digits of n. Let a be the low digit of n and c be the low digit of m so that

$$n = 3n' + a \qquad m = 3m' + c$$

From the low digit point formula (14), a and c are related by

$$\omega_3^{\mathit{dir}(m^0)}.\mathit{digit}(c) \equiv \omega_3^{\mathit{dir}(n^0)}.\mathit{digit}(a) + \omega_3^{\mathit{dir}(n)}.\mathit{dz} \mod b \qquad (17)$$

$\omega_3 \equiv 1 \mod b$ so the factors of ω_3 can be ignored, leaving c determined by a and dz. New direction difference δ' is those two low digits dropped from (16)

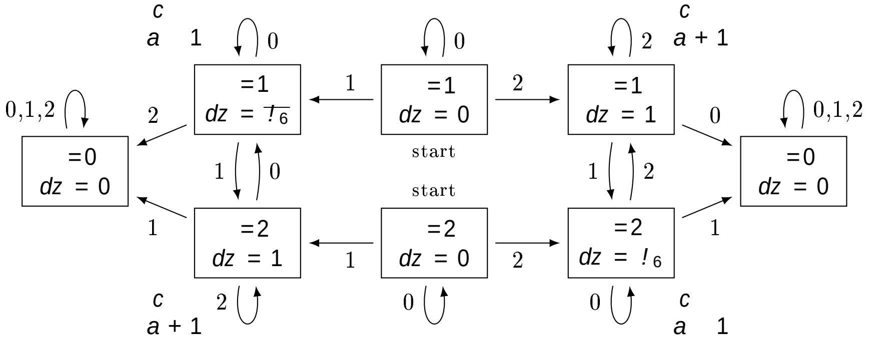
$$\delta' = \delta - \mathit{dir}(c) + \mathit{dir}(a)$$

New location offset is the low digits taken off (14). The whole m' is not known yet, but $\mathit{dir}(m') \equiv \mathit{dir}(n') + \delta' \mod 3$ is enough for its ω_3 power.

$$\begin{aligned}
\mathit{dz}'.\omega_3^{\mathit{dir}(n^0)} &= \mathit{point}(m') - \mathit{point}(n')\\
&= \quad \mathit{point}(m) - \omega_3^{\mathit{dir}(m^0)}.\mathit{digit}(c) \ /b\\
&\quad - \ \mathit{point}(n) - \omega_3^{\mathit{dir}(n^0)}.\mathit{digit}(a) \ /b\\
&= \ \mathit{dz}.\omega_3^{\mathit{dir}(n)} - \omega_3^{\mathit{dir}(n^0)+\delta^0}.\mathit{digit}(c) + \omega_3^{\mathit{dir}(n^0)}.\mathit{digit}(a) \ /b\\
\mathit{dz}' = & \ \ \mathit{dz}.\omega_3^{\mathit{dir}(a)} - \omega_3^{\delta^0}.\mathit{digit}(c) + \mathit{digit}(a) \ /b \qquad (18)
\end{aligned}$$

From (17) the bracketed part of (18) is a multiple of b.

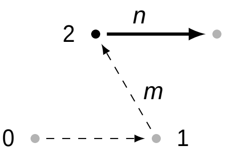
These steps begin from $\mathit{dz}=0$ so m and n are the same location, and $\delta=1$ or 2 other direction. The possible digits $a=0,1,2$ from n then give the following transitions between δ, dz combinations, and output digit c related to a. These are per figure 3.



$\delta=0, \mathit{dz}=0$ gives $c=a$ unchanged from there onwards.

A high 0 digit on n goes to state L2 or R2, or from R1 it goes to unchanged. The latter is when m is bigger than n, representing a further visit to the same location in a higher curve level.

In states L2 or R2, high 0 digits on n loop. To see the rule for these as adjacent arms, first for L2 suppose n had an extra high digit 2, so it goes to "unch", with new high $c=a-1=1$ on m.



So the other visit to n is at m along a curve directed from 1. Taking 2 as the origin means it is $3^k - m$ along a curve directed away from that 2, in arm -1 at $-60°$.

For R2 suppose n has an extra high digit 1, so it goes to "unch", with new high $c=a-1=0$. Taking 1 as the origin, this is m in the 0 curve which is $3^k - m$ away from 1 in arm 1 at $+60°$. □

L states are for n a left turn and R for n a right turn. They are reached from the S starts by lowest non-zero digit 1 or 2 respectively as per turn at (1).

Right boundary single-visited points are always left turns, otherwise non-overlapping plane filling would not be possible. So arm -1 is from R when high 0s on n don't reach "unch". Conversely left boundary points are right turns and arm $+1$ is from L. So the arm is either 0 when within the curve or $-\mathit{turn}(n)$ when adjacent arm.

The states of figure 3 and outputs can be expressed arithmetically using δ and the lowest non-zero digit of n,

$\mathit{other}(n, \delta)$ for $\delta = 1$ or 2

digits $n = a_k a_{k-1} \ldots a_0$ and extra high $a_{k+1} = 0$

```
output digits $c_{k+1}c_kc_{k-1}\dots c_0$
$a_t$ = lowest non-zero of $n$
$c_t\dots c_0 \leftarrow a_t\dots a_0$        unchanged
loop $j = t{+}1$ to $k{+}1$
    $c_j\ = 0,1,2\ \equiv (a_j - \delta.a_t) \bmod 3$        (19)
    $\delta\ \leftarrow\ \delta + \mathit{dir}(a_j) - \mathit{dir}(c_j)$        (20)
end loop
if $\delta \equiv 0 \bmod 3$ then $n_\delta =$              $c_{k+1}\dots c_0$, same arm
if $\delta \equiv 1 \bmod 3$ then $n_\delta = 3^{k+1} - c_{k+1}\dots c_0$, arm $-1$
if $\delta \equiv 2 \bmod 3$ then $n_\delta = 3^{k+1} - c_{k+1}\dots c_0$, arm $+1$
```

For δ at (20), taking dir of a single digit is simply 1 or 0 according as digit 1 or not. δ can be kept mod 3 at all stages.

a_t is the transition digit out of S states. Its use as $\delta.a_t$ at (19) flips the sense of δ for the R states. For example from S1 which is δ=1, an a_t=1 goes to L2 and a_t=2 goes to R1. Multiplying a_t gives $-\delta.a_t \equiv$ 2,1 to add for the output digit in those respective states.

The new n_δ can have up to 1 extra ternary digit over what n has. This is output digit c_{k+1} and the input a_{k+1} taken as 0.

If δ=0 is reached in the loop then all further digits are unchanged $c_j = a_j$. δ=0 means $c_j = a_j$ at (19) so $\mathit{dir}(c_j) - \mathit{dir}(a_j) = 0$ at (20), maintaining δ=0. If δ=0 initially then is no change $\mathit{other}(n, 0) = n$.

The L and R state δ, dz segments are located

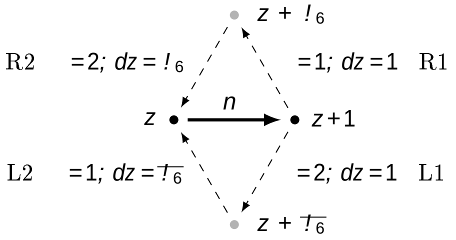


Figure 4: adjacent segments

Starting from these states gives, from n, the segment numbers of those segments. If in an adjacent arm then the reversal is $3^k{-}1 - \mathit{output}$ for segment rather than point.

Similar initial δ, dz can be used for other segments or points at further locations relative to n. Bigger dz may extend further than just one adjacent arm, going into other of the 6 arms which fill the plane.

other takes digits of n from low to high. Adjacent segments of (4) can also be high to low. Suppose a segment n has segment numbers s and e on its right. Expansion is a new low digit on n and the other segments

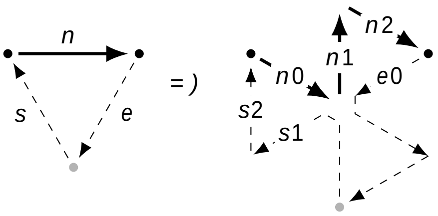


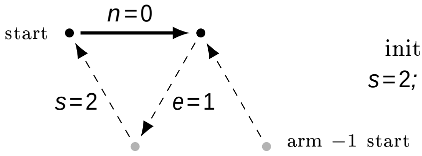
Figure 5: right side segment expansion

For example new low 0 on n means new adjacent segments are s with new low 2 or 1. The new segments from low of n are

n digit	s'	e'
0	$s2$	$s1$
1	$e0$	$n2$
2	$n1$	$e0$

(21)

Initial n=0 is no digits yet,



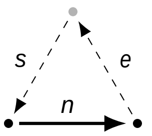
initial
s=2; e=1

Initial s=2, e=1 are segments in arm −1, on the right, directed towards the origin. Or instead start s=0 and an extra high 0 on n to step in (21) to 2, 1 (initial e being unused by this). A segment in arm −1 directed away from the origin is reversal 3^k-1- *output*. After all digits of n are processed an adjacent arm is identified by having high initial 1 or 2, above the digits of n.

other point of a left-turn n are given by one further low digit expansion. A further low 1 digit or 100…00 sequence on all of n, s, e are their middle common point. e is in direction δ=1 and s in direction δ=2. So for *other*(n) go high to low, not including the 1 which is lowest non-zero, and copy that 1 and low 0s to s and e.

Similar high to low holds for left side segments, and from them *other* of right turn n. The pattern of new low digits is the same as in (21), but which of n, s, e they take differs.

left
side
segments



n digit	s^0	e^0
0	$s2$	$n1$
1	$n0$	$s2$
2	$e1$	$e0$

(22)

In tables (21),(22), some entries copy n for the new s' or e'. This is where the output digits are to be n unchanged. It is somewhere at or above where the low to high of theorem 4 would be in "unch".

Theorem 5. $n-$ *other*(n, δ)
3

$$n - \textit{other}(n,\delta) \;=\; 3^{k_0} - 3^{k_1} + 3^{k_2} - \cdots + (-1)^t\, 3^{k_t} \qquad (23)$$
$$k_0 > k_1 > k_2 > \cdots > k_t \ge 1$$

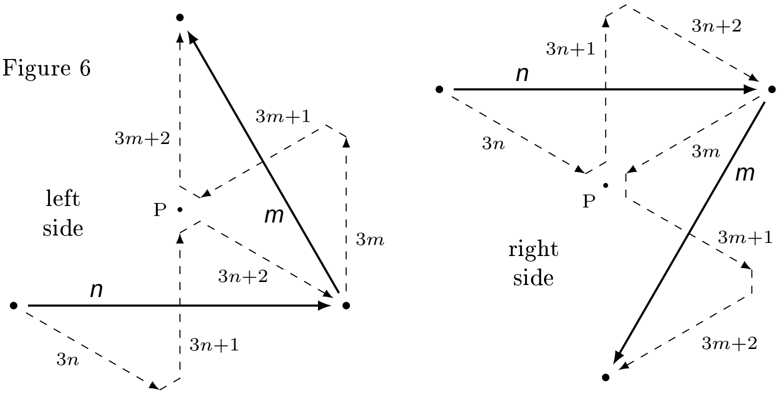
$$= 3, 6, 9, 18, 21, 24, 27, 54, 57, \dots$$

Firstly the claim will be that any pair of segments m, n around a unit triangle have difference $m{-}n$ of the following form and that in curve k all sums of this form with $p_0 < k$ occur.

$$\begin{aligned} m - n &= \pm 3^{p_0} \mp 3^{p_1} \pm \cdots + 1 \qquad p_0 > p_1 > \cdots > 0 && (24) \\ &\text{alternating signs, lowest term } +1 \\ &= 1, 7, 19, 25, 55, 61, 73, 79, \dots && \text{A055246} \\ &\ \ -2, -8, -20, -26, -56, -62, -74, -80, \dots && -\ \text{A190640} \end{aligned}$$

In $k{=}1$ this is true, with only difference $+1$ occurring. Expansion of a pair of sides is

Figure 6



Sides $3n, 3n{+}1$ is difference $+1$. Likewise $3n{+}1, 3n{+}2$, and likewise with m. The left triangle sides $3n{+}2, 3m$ are difference

$$3m - (3n{+}2) \ = 3(m{-}n) - 2$$

In (24), low term $+1$ becomes $+3$ then -2 reduces it to $+1$. So new difference is $m{-}n$ with the p powers incremented.

Sides $3m{+}1, 3n{+}2$ are difference

$$(3n{+}2) - (3m{+}1) \ = -3(m{-}n) + 1$$

-3 flips signs in $m{-}n$ so low term -3 which with further $+1$ is alternating. This is $m{-}n$ with all p powers incremented and new low -3.

These power increments with or without new low term give all forms in the new level.

In the manner of figure 5, or the new point P here in figure 6, expansion of sides of a unit triangle gives a new double or triple visited point. For segment n the new middle point on the right is $3n{+}1$. Two segment sides expanding to there are point difference

$$(3m{+}1) - (3n{+}1) = 3(m - n)$$

and thus the difference form (24). On further expansion the point visits are 3× each, so give any low k_t in (23). ☐

Differences can also be calculated from the *other* digit transformation of theorem 4. This shows where the difference powers fall in the *other* digit transformation.

The states of figure 3 loop on digit 0 or digit 2. For δ=1 the digit runs which loop and their resulting outputs are net ±1,

		or 1 go to unch L1	or 2 go to unch L2	
n	:::	022:::22	100:::00	10:::0
n_1	:::	100:::00	022:::22	10:::0
	+1	1	+1	

		or 1 go to unch R2	or 0 go to unch R1	
n	:::	200:::00	122:::22	20:::0
n_1	:::	122:::22	200:::00	20:::0
	+1	1	+1	

For δ=2 the runs are the same, but starting opposite lowest L1 and R2.
The states alternate and hence the signs for the increment. ☐

δ=1 can go to low run either R1 or L2, giving it either +1 or −1 lowest term. δ=2 low run R2 or L1 likewise. So δ=1 and δ=2 give the same set of differences.

turn = 1 goes to low R1,L1 always, but with an odd number of runs its highest can be −1 too and the absolute value flips all signs so that again *turn* = 1 or *turn* = −1 are the same set of differences.

In ternary, differences (23) are at least one low 0 digit, then an arbitrary digit, then digits 0 or 2 above. This is since each pair $3^{k_0} - 3^{k_1}$ is a run 022…22 and if the lowest term is t even then it is an unpaired $+3^{k_t}$ so lowest non-zero digit can be 1.

$$n \quad other(n;\ ) \; = \; \underbrace{\boxed{::: 0 \text{ or } 2 :::}}_{0 \text{ digits}}\boxed{\text{any}}\underbrace{\boxed{0:::0}}_{1 \text{ digits}} \tag{25}$$

$$= \text{ternary } 10, 20, 100, 200, 210, 220, 1000, 2000, 2010, \ldots$$

Theorem 6. d=0
k $d = 0, 1, 2$

$$S(k,d) = 3^{k-1} + s(k-4d).3^{\frac{k-1}{2}} \tag{26}$$

$$= \tfrac{1}{3}\ \ 3^k + \overline{\omega_3}^d b^k + \omega_3{}^d \bar{b}^k \tag{27}$$

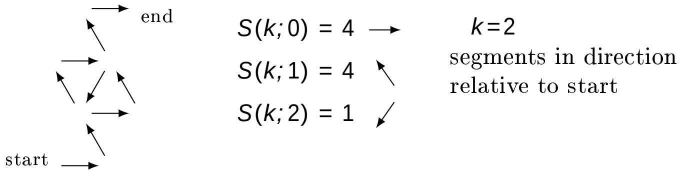
$$= \tfrac{1}{3}\ \ b^k + \omega_3^d{}^{\,2} - 1$$

$$s(j) = [2, 1, 1, 0, -1, -1, -2, -1, -1, 0, 1, 1] \qquad s(j-1) = \text{A214438}$$

$$S(k,0) = 1, 2, 4, \ \ 9, 24, 72, 225, 702, 2160, 6561, \ldots \qquad \text{A092236}$$

$$S(k,1) = 0, 1, 4, 12, 33, 90, 252, 729, 2160, 6480, \ldots \qquad \text{A135254}$$

$$S(k,2) = 0, 0, 1, \ \ 6, 24, 81, 252, 756, 2241, 6642, \ldots \qquad \text{A133474}$$



When the curve replicates the new sub-part 2 is in the same direction as the preceding level, so the segment counts double. The new sub-part 1 rotates +120°. The rotation means those segments in direction d=2 move to direction d=0. Similarly the other directions. So mutual recurrences

$$
\begin{aligned}
S(k{+}1,0) &= 2\,S(k,0) \;+\; S(k,2) && (28)\\
S(k{+}1,1) &= 2\,S(k,1) \;+\; S(k,0) && (29)\\
S(k{+}1,2) &= 2\,S(k,2) \;+\; S(k,1) && (30)
\end{aligned}
$$

Using (30) for $S(k,1)$ and substituting into (29) then using (28) for $S(k,2)$ and substituting again gives the following recurrence for d=0. By symmetry the same for d=1 and d=2.

$$S(k{+}3,d) = 6\,S(k{+}2,d) \;-\; 12\,S(k{+}1,d) \;+\; 9\,S(k,d)$$

The characteristic polynomial is

$$x^3{-}6x^2 + 12x - 9 \;=\; (x-3)(x-b)(x-\bar{b})$$

So $S(k,d)$ has a power form $X.3^k + Yb^k + Z\bar{b}^k$. From the initial values the coefficients are per (27).

The imaginary parts of the conjugate powers cancel out. Their real part gives factor $s(j)$ on the half power $3^{\lfloor (k-1)/2 \rfloor}$ for (26). ☐

There are 3^k segments in total. The selector function s has

$$s(j) + s(j{+}4) + s(j{+}8) \;=\; 0 \qquad \text{for all } j$$

so the half powers cancel out leaving

$$S(k,0) + S(k,1) + S(k,2) \;=\; 3^k$$

$S(k,d)$ can also be calculated by dir from (9). The segments in direction d=0 are those n which have $\mathrm{dir}(n) = 0,3,6,$etc. This means count $0,3,6,$etc many 1-digits among k ternary digits of n. The number of arrangements of those 1-digit positions is a binomial coefficient in k and then the remaining digits are each 0 or 2. So

$$
\begin{aligned}
S(k,0) &= 2^{k}\binom{k}{0} + 2^{k-3}\binom{k}{3} + 2^{k-6}\binom{k}{6} + \cdots\\
S(k,1) &= 2^{k-1}\binom{k}{1} + 2^{k-4}\binom{k}{4} + 2^{k-7}\binom{k}{7} + \cdots\\
S(k,2) &= 2^{k-2}\binom{k}{2} + 2^{k-5}\binom{k}{5} + 2^{k-8}\binom{k}{8} + \cdots\\
S(k,d) &= \sum_{j=d,d+3,\ldots} 2^{k-j}\binom{k}{j}
\end{aligned}
$$

These forms are among the power-weighted binomial sums considered by Justus[7] as a generalization of the binomial sums of Cournot and Ramus.

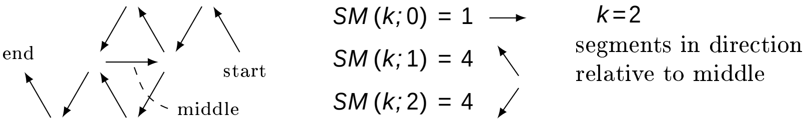
$S(k,0)$ was also a proposed International Mathematical Olympiad problem[6]. In that problem dividing out factors of 3 is ternary lowest non-0 which is the terdragon turn sequence. Summing is the direction $\mathit{dir}(n)$. Counting sums divisible by 3 is segments in direction $d{=}0$.

Theorem 7. $d = 0, 1, 2$

$$\begin{aligned}
\mathsf{SM}(k,d) &= S(k,\, d{+}k) \\
&= 3^{k-1} + \mathsf{sm}(k,d)\,.\,3^{\frac{k-1}{2}} \\
&= \tfrac{1}{3}\;\; 3^k + \omega_3^d(i\sqrt{3})^k + \overline{\omega_3^d(i\sqrt{3})^k} \\
&= \tfrac{1}{3}\;\;\; (i\sqrt{3})^k + \overline{\omega_3}^{d}\;{}^{2}\; - 1
\end{aligned}$$

$$\begin{aligned}
\mathsf{sm}(k,0) &= [\ \ 2,\ \ \ 0, -2,\ \ \ 0] \\
\mathsf{sm}(k,1) &= [-1, -1,\ \ \ 1,\ \ \ 1] \\
\mathsf{sm}(k,2) &= [-1,\ \ \ 1,\ \ \ 1, -1] \quad = \mathsf{sm}(k{+}1, 1)
\end{aligned}$$

$$\begin{aligned}
\mathsf{SM}(k,0) &= 1, 1, 1,\ \ 9, 33, 81, 225, 729, 2241, 6561, \ldots \qquad \text{A101990} \\
\mathsf{SM}(k,1) &= 0, 0, 4, 12, 24, 72, 252, 756, 2160, 6480, \ldots \\
\mathsf{SM}(k,2) &= 0, 2, 4,\ \ 6, 24, 90, 252, 702, 2160, 6642, \ldots
\end{aligned}$$



The middle segment is in direction $k \bmod 3$ so $\mathsf{SM}(k,d) = S(k, d{+}k)$. In $S(k, d{+}k)$ the factor $s(k - 4(d{+}k)) = s(-3k - 4d)$ gives $\mathsf{sm}(k,d)$. The $-3k \bmod 12$ becomes $k \bmod 4$ for $\mathsf{sm}(k,d)$. □

The periodic factors $\mathsf{sm}(k,d)$ can be expressed variously as powers of -1. For example $\mathsf{sm}(k,2) = (-1)^{\lfloor (k-1)/2 \rfloor}$ gives

$$\mathsf{SM}(k,2) = 3^{k-1} + (-3)^{\frac{k-1}{2}}$$

Theorem 8. $d{=}0$
n $d = 0, 1, 2$

$$\mathsf{SN}(n,d) = \tfrac{1}{3}\;\; n + 2\,\mathrm{Re}\,\overline{\omega_3}^{d}\mathit{point}(n) \tag{31}$$

$$\begin{aligned}
\mathsf{SN}(n,0) &= 0, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4, 4, 5, \ldots \\
\mathsf{SN}(n,1) &= 0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 6, \ldots \\
\mathsf{SN}(n,2) &= 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3, \ldots
\end{aligned}$$

There are total n segments,

$$\mathsf{SN}(n,0) + \mathsf{SN}(n,1) + \mathsf{SN}(n,2) = n \tag{32}$$

The real part of segments in direction 0 is +1 each. The real part of segments in directions 1 and 2 are $-\frac{1}{2}$ each. The total of these is net horizontal position *point*,

$$\mathsf{SN}(n,0) - \tfrac{1}{2}\mathsf{SN}(n,1) - \tfrac{1}{2}\mathsf{SN}(n,2) = \operatorname{Re}\mathsf{point}(n) \tag{33}$$

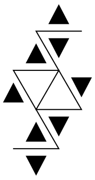
(32)+2×(33) cancels the direction 1 and 2 terms, giving the theorem for d=0. The other directions have corresponding forms after rotating by $\overline{\omega_3}$ or $\overline{\omega_3}^2$ so the desired d is the real part,

$$\begin{aligned} \mathsf{SN}(n,1) - \tfrac{1}{2}\mathsf{SN}(n,0) - \tfrac{1}{2}\mathsf{SN}(n,2) &= \operatorname{Re}\,\overline{\omega_3}\,\mathsf{point}(n) \\ \mathsf{SN}(n,2) - \tfrac{1}{2}\mathsf{SN}(n,0) - \tfrac{1}{2}\mathsf{SN}(n,1) &= \operatorname{Re}\,\overline{\omega_3}^2\,\mathsf{point}(n) \end{aligned}$$

Each combined with (32) gives the general case (31). ☐

2 Boundary

A unit triangle can be placed on each boundary segment of the curve. When the curve has a “V” notch a single triangle is placed in that notch touching both boundary segments.



These boundary triangles are similar in style to the boundary squares which Daykin and Tucker[5] count on the Heighway/Harter dragon curve.

Theorem 9.
k

$$\mathsf{BT}_k = 2^{k+1}$$

$$\mathsf{RT}_k = \mathsf{BT}_k/2 \;=\; 2^k \tag{34}$$

$$\mathsf{VT}_k = \mathsf{RT}_k$$

The “V” part boundary is between two level k curves at a 60° angle as in the following diagram. A level k curve can be drawn across the V endpoints to make a triangle.

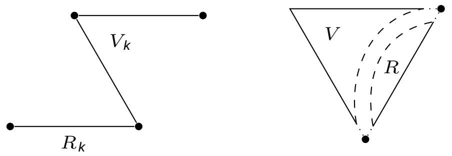


Figure 7:
R,V boundary parts
and triangle

Per plane filling theorem 2, all segments within the triangle are traversed precisely once so the unit triangles on the R boundary and those on the V boundary are identical $\mathit{VT}_k = \mathit{RT}_k$.

The left diagram shows that R_{k+1} comprises an R_k and a V_k. They meet as the outside of a 60° angle so do not have any boundary triangles in common.

$$\mathit{RT}_{k+1} = \mathit{RT}_k + \mathit{VT}_k \;=\; 2\,\mathit{RT}_k \tag{35}$$

Starting from $\mathit{RT}_0 = 1$ gives $\mathit{RT}_k = 2^k$. □

Each boundary triangle touches either 1 or 2 boundary segments. The two can be counted separately. The total is BT_k,

$$\mathit{BT}_k = \mathit{BT1}_k \;+\; \mathit{BT2}_k$$

Theorem 10.

$k \geq 1$

$$\mathit{BT1}_k = \begin{cases} 2 & k = 0 \\ \mathit{BT}_k/2 = 2^k & k \geq 1 \end{cases} \tag{36}$$

$$= 2, 2, 4, 8, 16, \dots$$

$$\mathit{BT2}_k = \begin{cases} 0 & k = 0 \\ \mathit{BT}_k/2 = 2^k & k \geq 1 \end{cases} \tag{37}$$

$$= 0, 2, 4, 8, 16, 32, \dots$$ A155559

$$\mathit{RT1}_k = \tfrac{1}{2}\mathit{BT1}_k = 1, 1, 2, 4, 8, 16, \dots$$ A011782

$$\mathit{RT2}_k = \tfrac{1}{2}\mathit{BT2}_k = 0, 1, 2, 4, 8, 16, \dots$$ A131577

$$\mathit{VT1}_k = \mathit{RT2}_k \qquad 1 \leftrightarrow 2$$

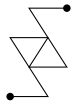
$$\mathit{VT2}_k = \mathit{RT1}_k$$

For k=1 the R boundary is two triangles, a 1-side and a 2-side, so they alternate.

Per the triangle of figure 7, the V boundary is the opposite side of an R, so each 1-side triangle of R is a 2-side triangle of V and vice-versa. These V triangles are in reverse order to R, so they are 1-side and 2-side alternately the same as R.

Level k+1 is an R_k followed by V_k and so alternates. □

The boundary of the curve can be measured by unit line segments around the outside of the curve.



k=2 boundary
B_2 = 12
R_2 = B_2=2 = 6

The boundary on one side is counted from start to end. The full boundary is counted by continuing around to the origin again.

The ends of the curve are isolated line segments (see theorem 16 for more on this). For the full boundary both the left and right sides of those ends are counted.

Theorem 11. k

$$B_k = \begin{cases} 2 & k = 0 \\ 3.2^k & k \geq 1 \end{cases} \tag{38}$$
$$= 2, 6, 12, 24, 48, 96, \ldots$$

$$R_k = B_k/2 = \begin{cases} 1 & k = 0 \\ 3.2^{k-1} & k \geq 1 \end{cases} \tag{39}$$
$$= 1, 3, 6, 12, 24, 48, \ldots$$

A003945

$$V_k = \begin{cases} 2 & k = 0 \\ 3.2^{k-1} & k \geq 1 \end{cases} \tag{40}$$
$$= 2, 3, 6, 12, 24, 48, \ldots$$

A042950

The boundary segments are found by counting the sides of the 1-side and 2-side boundary triangles (36),(37)

$$\begin{aligned} B_k &= BT1_k \;+\; 2BT2_k \\ R_k &= RT1_k \;+\; 2RT2_k \\ V_k &= VT1_k \;+\; 2VT2_k \end{aligned} \qquad \square$$

R and V parts expand as

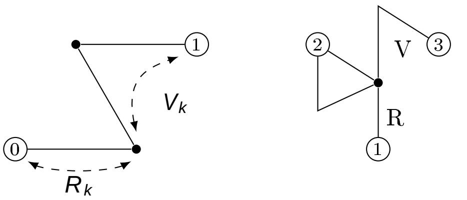


Figure 8:
R and V expansion,
initial segments
R_0 = 1
V_0 = 2

giving mutual recurrences

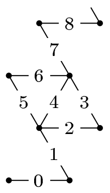
$$R_{k+1} = R_k + V_k \tag{41}$$
$$V_{k+1} = R_k + V_k \tag{42}$$

which are the same right-hand sides so $R_{k+1} = V_{k+1}$ and hence

$$\begin{aligned} R_{k+2} &= 2R_{k+1} && k \geq 0 \\ V_{k+2} &= 2V_{k+1} && k \geq 0 \end{aligned} \qquad \square$$

Recurrence (41) is the equivalent of (35) for the boundary triangles. (42) also holds for the boundary triangles per the expansion in figure 8, but doesn't show as clearly that the shape is opposite to R the way the triangle in figure 7 does.

$$\mathit{VT}_{k+1} = \mathit{RT}_k + \mathit{VT}_k$$



right boundary
segment numbers
0; 1; 2; 3; 7; 8; : : :

Theorem 12. $n{=}0$

$$\mathit{Rpred}(n) = \begin{cases} 1 & n \qquad 11 \quad 12 \quad 20 \\ 0 & n \end{cases} \tag{43}$$

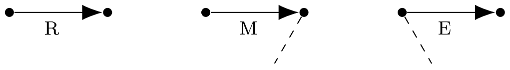
$$= 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, \dots$$

$$\begin{array}{ccccc} 0, 1, 2, & 3, 7, 8, & 9, 10, 11, & 21, 25, 26, & 27, 28, \dots \\ 0, 1, 2, & 10, 21, 22, & 100, 101, 102, & 210, 221, 222, & 1000, 1001, \dots \end{array}$$

n

= 1 at n = $\boxed{22{:}{:}{:}221}\boxed{00{:}{:}{:}00}\boxed{{:}{:}{:}}$ alternating runs, each run 1 digit

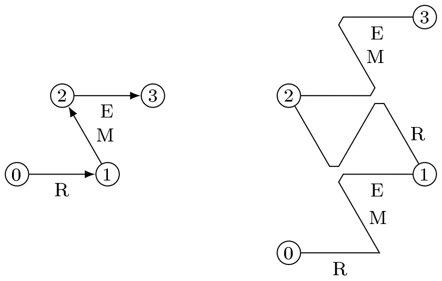
Take the boundary in three types of part



R has both endpoints on the boundary and is the right side of the full curve. M has an adjacent sub-curve at its end and so only some segments at its start are on the boundary. E has a sub-curve at its start.

Let R_k, M_k, E_k be the segment numbers which are on the boundary in the respective configurations at level k. These numbers are in the range 0 to $3^k - 1$ and hence can be written with k many ternary digits. The initial sets are a single 0 in each so $R_0 = M_0 = E_0 = 0$ corresponding to a single line segment. These zeros are understood as 0 many digits.

The curve expands as



The R segment 0–1 expands to sub-parts 0.R, 1.M, 2.E. The number 0, 1, 2 is the high ternary digit on top of the digits of the subsection. Treating each section this way gives

$$\begin{array}{rlll} R_k = & 0.R_{k-1}, & 1.M_{k-1}, & 2.E_{k-1} \\ M_k = & 0.R_{k-1} & & \\ E_k = & & 1.M_{k-1}, & 2.E_{k-1} \end{array}$$

Taking ternary digits from high to low, this expansion is a state machine. In state R any digit is permitted and switch to state R, M, E according to that digit. In state M only 0 is allowed and switch to state R. In state E either 1 or 2 is allowed and switch to state M or E.

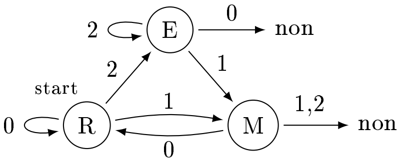


Figure 9: *Rpred*(*n*) state machine, ternary high to low

Digit 0, when permitted, always goes to state R. Digit 1 always goes to state M. Digit 2 always goes to state E. This means the state at any position is given by the preceding higher digit. A state transition permitted or not is therefore a digit pair permitted or not. So 11, 12, 20 not permitted. ☐

The lengths of sub-parts M and E are

$$M_k = \begin{cases} 1 & \text{if } k = 0, 1 \\ 3.2^{k-2} & \text{if } k \geq 2 \end{cases} \qquad \text{“M” part boundary length}$$

$$= 1, 1, 3, 6, 12, 24, 48, 96, \ldots$$

$$E_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ 3.2^{k-2} & \text{if } k \geq 2 \end{cases} \qquad \text{“E” part boundary length}$$

$$= 1, 2, 3, 6, 12, 24, 48, 96, \ldots$$

by writing the expansions as recurrences, initial $M_0 = E_0 = 1$, and substituting

$$\begin{aligned} R_{k+1} &= R_k + M_k + E_k \\ M_{k+1} &= R_k \\ E_{k+1} &= \qquad M_k + E_k \end{aligned}$$

M and E together are the V part $M_k + E_k = V_k$.

The states also give a count of how many sides the triangle on the right of segment n has. This is 1 or 2 for a boundary segment, or 3 for a non-boundary.

$$\mathit{Rsides}(n) = \begin{cases} 1 & \text{if } \mathit{Rpred} \text{ state R} \\ 2 & \text{if } \mathit{Rpred} \text{ state M or E} \\ 3 & \text{if } \mathit{Rpred} \text{ state ``non''} \end{cases} \qquad \text{right triangle sides} \tag{44}$$

$$= 3 - \mathit{Rpred}(n).[2,1,1] \tag{45}$$

$$= 1,2,2,1,3,3,3,2,2,1,2,2,3,3,3,3,3,3,3,3,3,1,\ldots$$

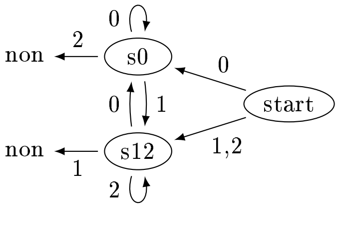
For (45), low 0 on n goes to state R and low 1 or 2 to states M,E, so n mod 3 determines respective factor 2 or 1 on *Rpred* to reduce from 3 sides.

Total *Rsides* in a level is 1 for each *RT1* triangle, 2 for each of the 2 segments of *RT2*, and 3 for each of the 3 segments of enclosed *AR* (ahead in section 3),

$$\sum_{n=0}^{3^k-1} \mathit{Rsides}(n) = \mathit{RT1}_k + 2.2.\mathit{RT2}_k + 3.3.\mathit{AR}_k \quad = \mathit{AR}_{k+2} \tag{46}$$

The geometric interpretation of total AR_{k+2} is that each respective $1,2,3$ side triangle after 2 expansions has $1,4,9$ unit triangles enclosed on the right, which are the coefficients 1, 2.2, 3.3 in (46).

A state machine for *Rpred* on ternary digits low to high follows by usual state machine manipulations reversing the high to low, or just from the allowed and disallowed digit pairs. State s0 is when the digit immediately below is a 0. State s12 is when the digit immediately below is 1 or 2.



Rpred(n) state machine,
ternary low to high

Theorem 13. *Rn*(m) $m \geq 0$

m

$m \leq 2$

m =	1	0 or 1	: : :	0 or 1	0; 1; 2

1, 2,

The effect of the change rule is that each maximal run $1,1,\ldots,1,\mathrm{NZ}$ becomes $2,2,\ldots,2,\mathrm{NZ}$, where NZ is a non-zero digit. If NZ is within the binary digits then it is 1. If NZ is the low ternary digit then it can be 1 or 2. In both cases its value is unchanged.

The allowed digit pairs are those not disallowed in theorem 12,

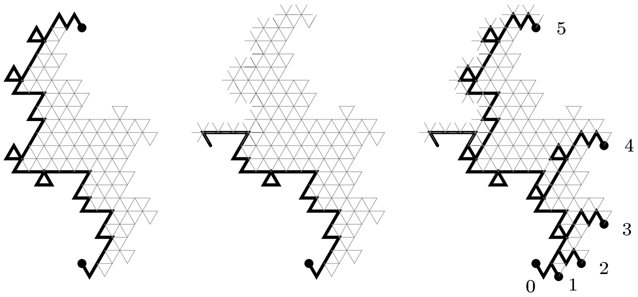
10	00
21	01
22	02

In a pair with a given low digit there are two choices for its high digit. For example 0 can have above it either 1 or 0 (the first row of the table). Start from low digit any $0, 1, 2$. Above it taking each of the two choices in the table steps through all and only allowed pairs. The highest digit must be non-zero and so the top-most pair is a single choice from the high 1-bit of the mixed base representation. ☐

Some of the left boundary in level k is enclosed by level $k{+}1$ and so is no longer on the boundary. (Unlike the right boundary which is never enclosed and so its level k boundary segment numbers are a prefix of the level $k{+}1$ boundary segment numbers.)

Three forms of left boundary segment numbers can be considered

- segments on boundary for particular level k
- segments on boundary for every level, so the curve continued infinitely
- segments on boundary for some level, a union of all left boundaries



left boundary level $k = 5$

left boundary infinite curve

left boundary all levels

Theorem 14. 0

02 10 11

k k 0

2 2

$k{-}1$

$$\begin{aligned} \mathit{Lpred}_k(n) &= \quad 02,10,11 \qquad k \qquad n \\ &= \mathit{Rpred}(3^k{-}1 - n) \\ &= 1 \qquad k{=}0 \\ &\quad\; 1,1,1 \qquad k{=}1 \\ &\quad\; 1,1,0,0,0,1,1,1,1 \qquad k{=}2 \end{aligned}$$

0

0 2

$$\begin{aligned} \mathit{Lpred}_\infty(n) &= \quad 02, 10, 11 \quad n \\ &= \mathit{Lpred}_k(n) \qquad 3^k{-}1 \geq 3n \\ &= 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, \ldots \end{aligned}$$

$$1 \quad n = \quad \begin{array}{l} \text{0,1, 5, 15,16,17, 45,46,50,51,52,53,...} \\ \text{0,1, 12, 120,121,122, 1200,1201,1212,1220,1221,1222,...} \end{array}$$

0

$$\begin{aligned} \mathit{Lpred}_{all}(n) &= \quad \mathit{Lpred}_k(n) \qquad k \qquad 3^k > n \\ &= 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, \ldots \end{aligned}$$

$$1 \quad n = \quad \begin{array}{l} \text{0,1,2, 5,6,7,8, 15,16,17,18,19,23,24,25,26,...} \\ \text{0,1,2, 12,20,21,22, 120,121,122,200,201,212,220,221,222,...} \end{array}$$

The curve is symmetric on its left and right sides, so the left boundary segment numbers are the right segment numbers but numbered in reverse $3^k{-}1{-}n$. This means digits 0,1,2 become 2,1,0. The digit pairs to exclude are the digit reversals of those in the right boundary pairs.

For the curve to level k the reversal is from endpoint $3^k{-}1$ and therefore applied to k digits.

For the curve extended infinitely the sub-part 2 is enclosed by the continuing curve, so the high digit cannot be 2, only 1.

For the union of all levels the reversal is made from any endpoint $3^k - 1 \geq n$. The endpoint giving no high 0 digits is the minimum disallowing. ☐

The number of sides on the triangle to the left of segment n follows in a similar way as a reversal of *Rsides* within k.

$$\begin{aligned} \mathit{Lsides}_k(n) &= \mathit{Rsides}(3^k{-}1 - n) \qquad \text{left triangle sides} \\ &= 1 \quad \text{for } k{=}0 \\ &\quad 2, 2, 1 \quad \text{for } k{=}1 \\ &\quad 2, 2, 3, 3, 3, 1, 2, 2, 1 \quad \text{for } k{=}2 \\ \mathit{Lsides}_\infty(n) &= \mathit{Lsides}_k(n) \quad \text{for } 3^k > 3n \\ &= 2, 2, 3, 3, 3, 1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 1, 3, 3, 3, 3, \ldots \end{aligned}$$

Theorem 15. $\mathit{Ln}(m)$ $m \geq 0$

m

k k

m =

0 or 1	0 or 1	: : :	0 or 1	0; 1; 2

k digits

0

m =

0	1	0 or 1	: : :	0 or 1	0; 1; 2

$m \le 2$ $Ln(m) = m$ $m=3$ $Ln(3)$
$=5$ $m \ge 4$ $m+2$
$10 \to 1$ 1 $11 \to 01$

$m+2 =$

1 or 01	0 or 1	: : :	0 or 1	0; 1; 2

digit below	bit 0	bit 1
0	0	2
1	0	2
2	1	2

$Ln(m)$

For example for the infinite curve $m=8$ is mixed radix 0102. The low 0 has a 2 below it so per the third row of the table that bit 0 changes to a 1 digit 0112. Then the next higher position is a 1 bit and the digit below is 1 so per the second row of the table change that bit 1 to digit 2 giving 0212. Finally the high 0 has a 2 below so per the third row of the table that bit 0 changes to digit 1 for final ternary 1212 = decimal 50. This is the $m=8$ sample value shown in theorem 14 (the first value as $m=0$).

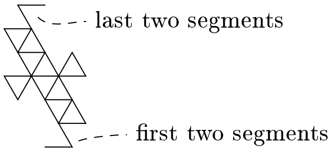
The allowed digit pairs for the left boundary are those not disallowed in theorem 14. The transformations give all and only these pairs.

20	00
21	01
22	12

For the curve extending infinitely the extra high 0 bit ensures the high ternary digit is not 2 since the third row of the table transforms 0→1 when a 2 is below (and leaves 0 unchanged for digit 0 or 1 below).

For the union of all levels the mixed radix forms are to be those of all k. When there is one high 0 bit it becomes either 0 or 1 per the bit 0 column of the table. Any further high 0 bits would remain as 0, per the first two rows of the bit 0 column. Therefore the values resulting from two or more high 0s are the same as from a single high 0. So it suffices to take mixed forms with and without a single extra 0 bit. The rule in the theorem uses the second highest bit to choose with or without. The mixed radix is formed on $m+2$ since there are just 4 initial values 0,1,2,5 before beginning this mixed form. ☐

Theorem 16. k



For k=0 the single segment is on the left and right boundary.

For k=1 the three segments $0, 1, 2$ are on the left and right boundary.

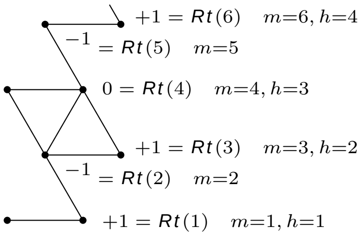
For $k \geq 2$, combining digit pair conditions of theorem 12 and theorem 14 gives permitted digit pairs only $00, 01, 21, 22$ for segment on both left and right boundaries. The only numbers which can be made with these pairs are

0 1	first two segments
222 . . . 221 222 . . . 222	last two segments

These are the first two and last two segments. For two digits they are simply the four permitted pairs. □

The right boundary of the terdragon at each point turns either +120° (left), −120° (right), or goes straight ahead. Number the right boundary points starting from m=0 so the first turn is at m=1.

The following diagram illustrates the first few boundary turns.



Theorem 17.

-1

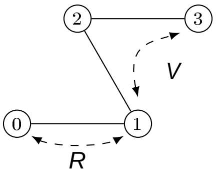
$$Rt(m) = \begin{cases} -1 & m \equiv 2 \bmod 3 \\ +1 & m \not\equiv 2 \bmod 3 \quad \mathit{BitAboveLowestOne}(h) = 0 \\ 0 & m \not\equiv 2 \bmod 3 \quad \mathit{BitAboveLowestOne}(h) = 1 \end{cases}$$

$$h = m - \lfloor m/3 \rfloor \qquad -1$$

$$= +1,\ -1,\ +1,\ 0,\ -1,\ +1,\ +1,\ -1,\ 0,\ 0,\ -1,\ +1,\ +1,\ -1,\ \ldots$$

$$\mathit{BitAboveLowestOne}(h) = \qquad h$$

Take the curve boundary in two parts R and V



Initial turns
R_0 = empty
V_0 = 1 (right)

The turn at 1 is always left, so

$$R_{k+1} = R_k,\ +1,\ V_k$$

As per figure 8, V_{k+1} is an R and V with 0° turn (straight ahead) in between,

$$V_{k+1} = R_k,\ 0,\ V_k$$

These expansion rules are the dragon curve turn sequence, and per Davis and Knuth[3] those turns are bit-above-lowest-1. The initial R_0 = empty and $V_0 = -1$ mean the final V expansion adds an extra -1 at every third position starting from m=2. □

3 Area

The area enclosed by the curve can be counted in unit triangles. The curve does not cross itself so each enclosed triangle is either on the left or the right side of the curve.

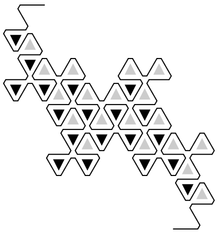


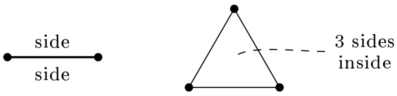
Figure 10:
k=4 enclosed area
black right of curve
gray left of curve
AL_4 = AR_5 = 19
total A_4 = 38

The left and right side triangles alternate along each row and each diagonal. The left side is all the upward pointing triangles. The right side is all the downward pointing triangles. (This arises later in theorem 21 with the Cantor dust.)

Lemma 1. A B
N

$$3A + B = 2N \tag{47}$$

Count the sides of the line segments. There are N segments so total $2N$ sides. Each side is either on a boundary or is inside.



There are B outside sides on the boundary. The inside sides are all in enclosed unit triangles. Each area triangle A has 3 inside sides, so $3A$ inside sides and total $B + 3A = 2N$. ☐

Theorem 18. k

$$A_k = \begin{cases} 0 & k = 0 \\ 2\;3^{k-1} - 2^{k-1} & k \geq 1 \end{cases} \tag{48}$$

$$= 0, 0, 2, 10, 38, 130, 422, 1330, 4118, \ldots \qquad {\scriptstyle k\geq 1}\ \text{A056182}$$

$$\begin{aligned} \mathit{AR}_k = \mathit{AL}_k &= A_k/2 \\ &= \begin{cases} 0 & k = 0 \\ 3^{k-1} - 2^{k-1} & k \geq 1 \end{cases} \\ &= 0, 0, 1, 5, 19, 65, 211, 665, 2059, \ldots \qquad {\scriptstyle k\geq 1}\ \text{A001047} \end{aligned}$$

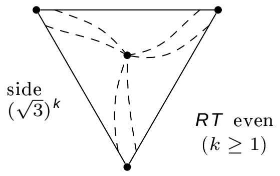
Non-crossing theorem 1 and plane filling theorem 2 mean that for all lengths every enclosed unit triangle has all three sides traversed. If this were not so then the curve would have to cross itself, or another copy of the curve cross in, to fill that area to make 6 copies plane filling.

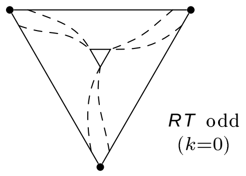
So lemma 1 applies with $N = 3^k$ line segments and boundary B_k from (38).

$$\begin{aligned} 3A_0 + 2 &= 2.3^0 && \text{for } k = 0 \\ 3A_k + 3.2^k &= 2.3^k && \text{for } k \geq 1 \end{aligned}$$

Non-crossing means each enclosed unit triangle is either on the left or right side of the curve. By symmetry the two sides are equal so half the area each. ☐

When three terdragon curves are arranged in a triangle all segments inside are traversed precisely once (by non-crossing plane filling again) so the unit triangles are either enclosed by one side of the curve or are boundary triangles. The boundary triangles from the three curves overlap as in the following diagram.





Boundary triangles of adjacent sides overlap. If RT_k is even then by symmetry there is a vertex in the middle common to all three. If RT_k is odd then there is a unit triangle in the middle which is common to all three.

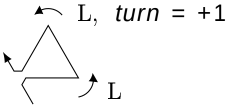
The curve length end-to-end is $(\sqrt{3})^k$ and triangles of curves like this partition the plane into identical shapes so there are 3^k unit triangles inside.

$$\begin{aligned} 3^k &= 3\,\mathit{AR}_k + 3\,\mathit{RT}_k/2 && \text{if } \mathit{RT}_k \text{ even} \\ 3^k &= 3\,\mathit{AR}_k + 3\,(\mathit{RT}_k - 1)/2 + 1 && \text{if } \mathit{RT}_k \text{ odd} \end{aligned} \tag{49}$$

☐

RT_k from (34) is odd only for k=0. When RT_k is even, (49) is equivalent to $3A+B = 2N$ from (47). The boundary triangles alternate 1-side and 2-side from theorem 10 giving $R_k = \frac{3}{2}\mathit{RT}_k$ for $k \ge 1$, so that (49) is $3^k = 3A_k/2 + B_k/2$.

As from *TurnRun* in subsection 1.2, the curve turns go in runs of either 1 or 2 consecutive left or right. A run of 2 consecutive turns encloses a unit triangle.

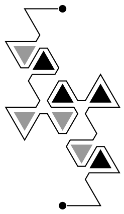


2 consecutive left turns,
is left-side enclosed unit triangle

The run lengths are pairs either 1,2 or 2,1. There is one 2 for each of the $3^{k-1}-1$ turns of the previous expansion level. So the number of runs of 2 turns in curve k is

$$\mathit{TurnRuns2}_k = \begin{cases} 0 & \text{if } k{=}0 \\ 3^{k-1} - 1 & \text{if } k \ge 1 \end{cases}$$

$$= 0, 0, 2, 8, 26, 80, 242, \ldots \qquad k{\ge}1 \text{ A024023}$$



k=3
LL squares black
RR squares gray

total
$\mathit{TurnRuns2}_3$ = 8
A_3 = 10

The proportion of enclosed unit triangles formed by 2-turns, out of the total area, is

$$\frac{\mathit{TurnRuns2}_k}{A_k} = \frac{1}{2} + \frac{2^{k-1}-1}{A_k} \to \frac{1}{2}$$

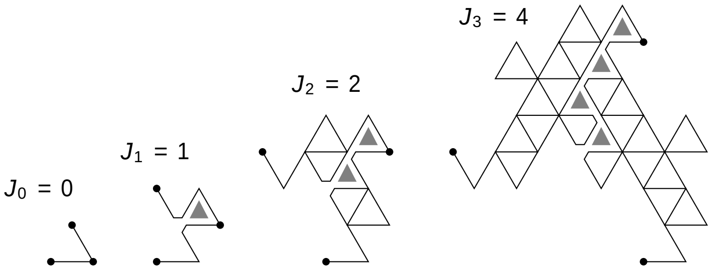
This limit is approached from above since $2^k-1 > 0$ for $k \ge 2$ which is where $A_k > 0$. For example in k=3 the ratio is $\frac{4}{5}$,

Some segments have these triangles on both sides. Such pairs are a sequence of turns LLRR. As from the turn expansion in figure 2, such consecutive 2-runs occur only as an LR with L,R existing turns surrounding. An L,R is then only the middle of an LLRR of preceding segment expansion. So there is one LLRR for each $k-2$ segment.

There are no RRLL pairs, since the Rs could only be an LRR with existing R, but then LR follows, not LL.

$$\mathit{TurnRuns2pairs}_k = \begin{cases} 0 & \text{if } k = 0, 1 \\ 3^{k-2} & \text{if } k \ge 2 \end{cases}$$

The join between two terdragon curves at 60° angle encloses new area.

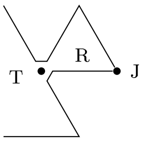


Theorem 19. k

$$J_k = \begin{cases} 0 & k = 0 \\ RT_{k-1} & k \geq 1 \end{cases}$$

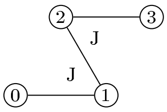
$$= 0, 1, 2, 4, 8, 16, 32, 64, \ldots \qquad \text{A131577}$$

Two curves $k \geq 1$ have their $k{-}1$ sub-curves touching at point T as follows.



T is on the boundary since there are two absent sub-curves there (West and South-West). The join start J through to T is a curve side so the join area is its right boundary triangles RT_{k-1}. ☐

Join area can also be calculated from the excess of area A_{k+1} over three copies of the previous A_k. This counts the join triangles but doesn't give their shape.



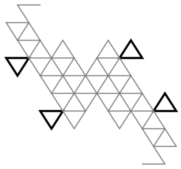
One join area is on the left side of the curve and one is on the right. The curve is symmetric left and right so the two joins are the same size.

$$A_{k+1} - 3A_k = 2J_k$$

The joins are also the shortfall of the boundary B_{k+1} over three copies of the previous B_k. Each unit triangle enclosed by the joins reduces the boundary by 3 segments,

$$3B_k - B_{k+1} = 2\,.\,3\ J_k$$

On the boundary of the terdragon curve there are some hanging unit triangles which touch the rest of the curve at only a single point.



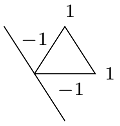
k=4
hanging triangles
$H_4 = 4$
$HR_4 = H_4/2 = 2$

Theorem 20. k

$$H_k = \begin{cases} 0 & k = 0,1,2 \\ 2^{k-2} & k \geq 3 \end{cases}$$
$$= 0,0,0,2,4,8,16,32,\dots$$

$$HR_k = \tfrac{1}{2} H_k = \begin{cases} 0 & k = 0,1,2 \\ 2^{k-3} & k \geq 3 \end{cases}$$
$$= 0,0,0,1,2,4,8,16,\dots$$

A hanging triangle is boundary turn sequence $-1,1,1,-1$ as from subsection 2.4.



This is a pair $\mathit{BitAboveLowestOne}(j) = 0$ and $\mathit{BitAboveLowestOne}(j{+}1) = 0$ with j even. This requires j is binary low 0100, and possible further low 0 bits.

$$j = \begin{array}{|c|c|c|c|c|c|}\hline \text{any} & 0 & 1 & 0 & 0 & \underbrace{0\dots0}_{\geq 0 \text{ zeros}} \\ \hline \end{array} \qquad \text{total } k \text{ bits}$$

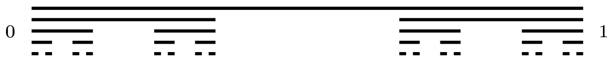
The "any" bits at the high end can be any value of length 0 to $k-4$ bits. In addition the "1" shown can be the highest bit for value $j = 100\dots00$ binary. The total number of such values is therefore

$$HR_k = 1 + \sum_{i=0}^{k-4} 2^i = 2^{k-3} \qquad \text{for } k \geq 3$$

For k=3 the sum is understood as empty so $HR_3 = 1$ which is single value $j = 100$ in binary. When $k \leq 2$ there are not enough bits to have any "100" at all and so $HR = 0$. □

4 Cantor Dust

The Cantor dust fractal is formed by removing the middle third of a line segment and doing the same to each remaining line segment recursively.



An integer version can be formed by multiplying by 3^k. The effect is to start with a unit line segment and triple out by a gap then a copy.

Counting the first segment as 0, segment number n is present or not according to

$$\mathit{Cpred}(n) = \begin{cases} 1 & \text{if } n \text{ in ternary has no 1 digits} \\ 0 & \text{if } n \text{ in ternary has one or more 1 digits} \end{cases}$$

$$= 1,0,1,0,0,0,1,0,1,\ 0,0,0,0,0,0,0,0,0,\ 1,0,1,0,0,0,1,0,1 \qquad \text{A088917}$$

Theorem 21.

Let *Tperm* change ternary digit pairs 10 to 20 and vice versa 20 to 10. This is a self-inverse permutation of the integers.

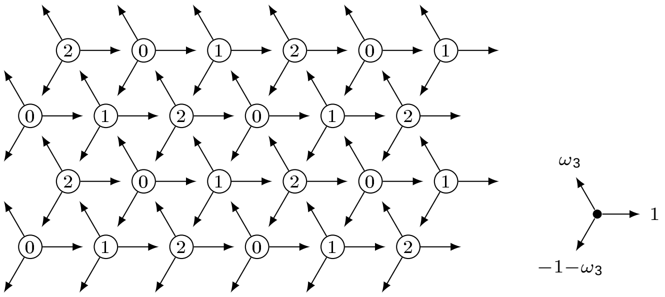
$$\begin{aligned} \mathit{Tperm}(n) &= \text{flip ternary digit pairs } 10 \leftrightarrow 20 \text{ of } n \\ &= 0,1,2,6,4,5,3,7,8,18,19,20,15,13,14,12,16,\ldots \\ \text{ternary} &= 0,1,2,20,11,12,10,21,22,\ 200,201,202,120,111,\ldots \end{aligned}$$

In *Rpred* (43), with *Tperm* applied the digit pairs 10 allowed and 20 disallowed become instead 10 disallowed and 20 allowed. So $\mathit{Rpred}(\mathit{Tperm}(n))$ has pairs 10, 11, 12 disallowed and hence

$$\mathit{Cpred}(n) = \mathit{Rpred}(\mathit{Tperm}(3n)) \tag{50}$$

The terdragon right boundary segments occur in triplets which have successively $n \equiv 0,1,2 \bmod 3$ (since any non-boundary excursion is a multiple of 3 length). A Cantor unit segment is identified with such a triplet.

For the enclosed unit triangles, the terdragon curve always steps in direction 0°, 120° or −120°. Any path taking such steps has each unit triangle with segment numbers going $0,1,2 \bmod 3$ in the following pattern.



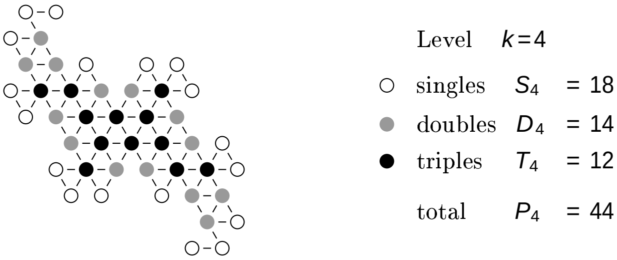
For a point at $x+y\omega_3$ the number shown is $x+y$ mod 3. Stepping in direction 1, ω_3 or $-1-\omega_3$ which are 0°, 120° or −120° change that $x+y$ index by +1 mod 3. Hence the pattern.

Each unit triangle is either on the left or right side of each segment. Those on the left have segment numbers going clockwise. Those on the right have segment numbers going anti-clockwise.

The right-side unit triangles are all the right-side non-boundary segments. Each unit triangle can be identified by its 0 mod 3 segment and this corresponds to the Cantor non-segments as per (50). ☐

5 Points

The terdragon curve touches at various vertices. Each point may be visited 1, 2 or 3 times.



Theorem 22.
k

$$S_k = \begin{cases} 2 & k=0 \\ 2^k+2 & k\geq 1 \end{cases} \tag{51}$$

$$= 2, 4, 6, 10, 18, 34, \ \ 66, 130, 258, \dots \qquad \text{A133140}$$

$$D_k = \begin{cases} 0 & k=0 \\ 2^k-2 & k\geq 1 \end{cases}$$

$$= 0, 0, 2, \ \ 6, 14, 30, \ \ 62, 126, 254, \dots$$

$$T_k = \begin{cases} 0 & k=0 \\ 3^{k-1}-2.2^{k-1}+1 & k\geq 1 \end{cases}$$

$$= 0, 0, 0, \ \ 2, 12, 50, 180, 602, 1932, \dots \qquad k{\geq}1 \ \ \text{A028243}$$

For k=0 the curve is a single line segment. Each end is a single-visited point.

For $k \geq 1$, when each line segment of the previous level expands it makes a new vertex in the middle of an adjacent triangle.

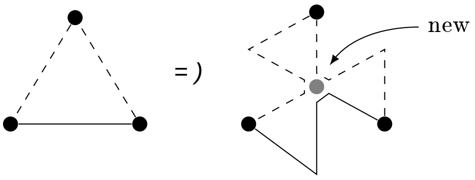


Figure 11:
new vertex
beside segment

The visits to the original vertex points are unchanged by the expansion. The visits to each new middle point are the number of sides of the triangle. Triangles with three sides are the enclosed area A_k (48). Each of them gives a new triple-visited point. Triangles with 1 or 2 sides are the boundary triangles $\mathsf{BT1}_k$ and $\mathsf{BT2}_k$ from (36),(37). Each of them gives a single or double visited point respectively. So the following recurrences, giving sums. The sums are taken as empty when k=0.

$$S_k = S_{k-1} + \mathsf{BT1}_{k-1} \quad = 2 + \sum_{j=0}^{k-1} \mathsf{BT1}_j$$

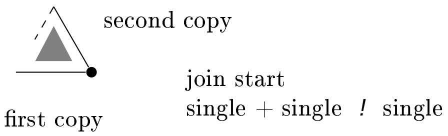
$$D_k = D_{k-1} + \mathsf{BT2}_{k-1} \quad = \sum_{j=0}^{k-1} \mathsf{BT2}_j$$

$$T_k = T_{k-1} + A_{k-1} \quad = \sum_{j=0}^{k-1} A_j$$

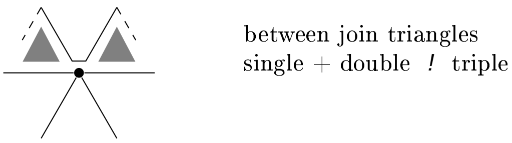
□

When the curve triples to make its next level there are three copies of the points. Where they join some point visits merge.

Each sub-curve endpoint is single-visited and when they join it remains a single,

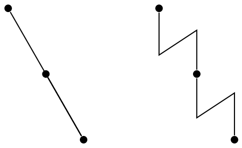


Adjacent join area triangles touch at a corner as follows.

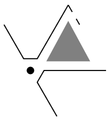


The join touches are always a single meeting a double since otherwise there would be untraversed segments within the curve.

The boundary at the end of a join is always a straight line. This is so for the first join in level k=2 and for any subsequent level the expansion is



A straight line at the join end can only be formed from two single-visited points becoming double-visited.



There are two identical join areas so the above merges apply twice. When there is at least one join triangle $J_{k-1} \geq 1$, which is when $k \geq 2$, the following recurrences

$$\begin{aligned} S_k &= 3\,S_{k-1} + 2 \ \ -(J_{k-1} - 1) - 3 && \text{for } k \geq 2 \\ D_k &= 3\,D_{k-1} + 2 \ \ -(J_{k-1} - 1) + 1 \\ T_k &= 3\,T_{k-1} + 2 \qquad J_{k-1} - 1 \end{aligned} \tag{52}$$

There are $J_{k-1} - 1$ new triple points in between join triangles. They reduce the singles and doubles and increase the triples. The singles are further -1 at the join start and -2 at the join end. The doubles are $+1$ at the join end. With $J_{k-1} = 2^{k-2}$ and the initial S,D,T values the formulas (51) etc follow. ☐

Per OEIS A028243, the triples T_k are twice Stirling numbers of the second kind

$$T_k = 2\,\mathit{St}(k, 3) \qquad \text{Stirling second kind}$$

Triples recurrence in J at (52) is the usual Stirling recurrence since $J_k - 1 = 2^{k-1} - 1 = \mathit{St}(k, 2)$ for $k \geq 1$.

$$\begin{aligned} T_k/2 &= 3T_{k-1}/2 \quad + \ J_{k-1} - 1 && \text{(52)/2, for } k \geq 2 \\ \mathit{St}(k, 3) &= 3\mathit{St}(k-1, 3) + \mathit{St}(k-1, 2) && \text{Stirling recurrence} \end{aligned}$$

All single and double visited points are on the boundary. Some triple visited points are on the boundary too. A boundary triple is in a V shape 2-side boundary triangle. The 4 such at curve start and end are not triple visited. At hanging triangles there is a V each side of a triple point.

$$\begin{aligned} \mathit{TB}_k &= \mathit{BT2}_k - H_k - 4 \qquad \text{for } k \geq 2 \\ &= \begin{cases} 0 & \text{if } k \leq 2 \\ 3.2^{k-2} - 4 & \text{if } k \geq 3 \end{cases} \qquad \text{triple-visited on boundary} \\ &= 0, 0, 0, 2, 8, 20, 44, 92, 188, \ldots && {\scriptstyle k\geq 3 \text{ A131128}} \end{aligned}$$

The total number of distinct visited points is

$$P_k = S_k + D_k + T_k$$

$$= \begin{cases} 2 & \text{if } k=0 \\ 3^{k-1}+2^k+1 & \text{if } k\geq 1 \end{cases} \quad \text{distinct points}$$

$$= 2, 4, 8, 18, 44, 114, 308, 858, \ldots \qquad {\scriptstyle k\geq 1} \quad 2{\times}\text{A099754}$$

It can be noticed

$$P_k + A_k = 3^k + 1$$

In general $P+A = N+1$ for any path with N line segments on a triangular grid which is non-overlapping and each enclosed unit triangle has all three sides traversed. Such a path starts as a single point and no line segments. Then each further line segment either goes to an unvisited point which increases P, or it revisits a point and encloses a new unit triangle which increases A. So for each N either A or P increments.

Per figure 11, the number of sides of the triangle adjacent to a segment determines the number of visits to new points $n \equiv 1,2 \bmod 3$. The number of visits is unchanged by further expansions, which are low ternary 0-digits.

$$\mathit{Visits}_k(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } 3^k \\ \mathit{Rsides}(n) & \text{if } n = (3m{+}1).3^l,\ \ m\geq 1 \\ \mathit{Lsides}_{k-l-1}(n) & \text{if } n = (3m{+}2).3^l \end{cases}$$

$$\begin{aligned} &= 1,1 \quad \text{for } k{=}0 \\ &\quad 1,1,1,1 \quad \text{for } k{=}1 \\ &\quad 1,1,2,1,2,2,1,2,1,1 \quad \text{for } k{=}2 \end{aligned}$$

For the curve continued infinitely, Lsides_∞ is used. Or it suffices to take 1 level bigger,

$$\begin{aligned} \mathit{Visits}_\infty(n) &= \mathit{Visits}_k(n) \qquad \text{for } 3^k > 3n \\ &= 1,1,2,1,2,2,2,2,3,1,1,3,2,3,3,2,3,1,2,3,\ldots \\ {=}1 \text{ at } n &= 0,\ \ 1,\ \ 3,\ \ 9,10,17,27,28,30,51,53,64,\ldots \\ {=}2 \text{ at } n &= 2,\ \ 4,\ \ 5,\ \ 6,\ \ 7,12,15,18,21,22,25,31,\ldots \\ {=}3 \text{ at } n &= 8,11,13,14,16,19,20,23,24,26,29,32,\ldots \end{aligned}$$

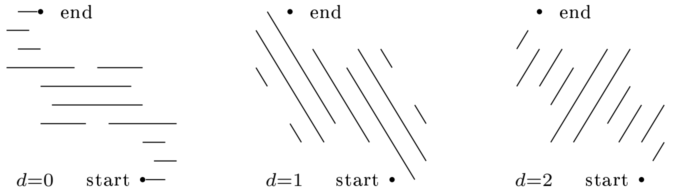
Visits also follow from $\mathit{other}(n,\delta)$ from theorem 4. The visits are all those occurring in the same curve arm and within the same k, or same arm and anywhere for the curve continued infinitely.

$$\begin{aligned} \mathit{Visits}_k(n) &= \operatorname*{count}_{\delta=0 \text{ to } 2} \ \mathit{other}(n,\delta) \text{ same arm and} \leq 3^k \qquad (53) \\ \mathit{Visits}_\infty(n) &= \operatorname*{count}_{\delta=0 \text{ to } 2} \ \mathit{other}(n,\delta) \text{ same arm} \end{aligned}$$

Total of Visits within level k is 1 for each single, 2 each for the 2 visits to doubles, and 3 each for the 3 visits to triples.

$$\begin{aligned} \sum_{n=0}^{3^k} \mathit{Visits}_k(n) &= S_k + 4D_k + 9T_k \quad = 3.3^k - 4.2^k + 3 \\ &= 2, 4, 14, 52, 182, 604, 1934, \ldots \qquad 2{\times}\text{A134063} \end{aligned}$$

Some segments in the terdragon are consecutive and they can be considered in runs making lines in directions $d = 0, 1, 2\ \times 120°$.



$Lines_4(0)$ = 12 $Lines_4(1)$ = 9 $Lines_4(2)$ = 10

k=4; total $Lines_4$ = 31

Theorem 23. k

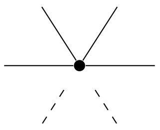
$$\begin{aligned} \mathit{Lines}_k &= 2^{k+1} - 1 \\ &= 1, 3, 7, 15, 31, 63, 127, 255, \ldots \end{aligned}$$

A126646

There are 3^k line segments. If none are consecutive then the segments are the lines. This occurs for k=0 and k=1 with $\mathit{Lines}_0 = 1$ and $\mathit{Lines}_1 = 3$.

At each triple-visited point there are consecutive line segments in all 3 directions, reducing the lines by 3.

Each double-visited point must have its two absent segments adjacent or the curve would cross or overlap when filling the plane.



double-visited point
missing segments are
adjacent around the point

So at each double-visited point there are consecutive line segments in one direction, reducing the lines by 1.

$$\mathit{Lines}_k = 3^k - 3\,T_k - D_k \tag{54}$$

□

A similar argument can be made counting line ends.

At a single visited point there are 2 line ends, except for the curve start and end where just 1 each, so $2S_k - 2$ ends from singles.

At a double-visited point there is one line continuing across and 2 lines ending.

At a triple-visited point there are no line ends (all 3 directions continue across).

Every line has 2 ends so

$$\mathit{Lines}_k = \tfrac{1}{2}\,(2S_k - 2\ +\ 2D_k) \tag{55}$$

□

Difference (54) − (55) is the total 3^k+1 visits to all points

$$3^k + 1 = S_k + 2\,D_k + 3\,T_k$$

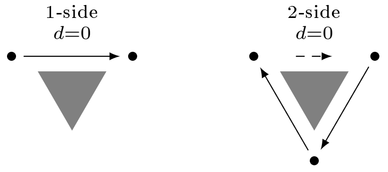
Theorem 24. $d = 0,1,2$ k

$$\begin{aligned}
\mathit{Lines}_k(0) &= \tfrac{1}{3}\ \ 2^{k+1} + \mathit{Ld}(k)\\
\mathit{Lines}_k(1) &= \tfrac{1}{3}\ \ 2^{k+1} - \mathit{Ld}(k-1)\\
\mathit{Lines}_k(2) &= \tfrac{1}{3}\ \ 2^{k+1} - \mathit{Ld}(k+1)\\
\mathit{Ld}(m) &= [1,2,4,5,4,2]_m
\end{aligned}$$

$$\begin{aligned}
\mathit{Lines}_k(0) &= 1,2,4,7,12,22,43,86,172,343,684,\ldots\\
\mathit{Lines}_k(1) &= 0,1,2,4,\ \ 9,20,42,85,170,340,681,\ldots\\
\mathit{Lines}_k(2) &= 0,0,1,4,10,21,42,84,169,340,682,\ldots && \text{A111927}
\end{aligned}$$

Lines in the three directions are each $\frac{1}{3}$ of the total except for the variation by Ld, giving differences up to 3, depending on k.

Use line ends similar to the second proof above, but with ends in each direction d. Start with boundary triangles. Count 1-side boundary triangles by the direction of their segment. Count 2-side boundary triangles by the direction of their missing segment.



1-side and 2-side
right boundary triangles
in direction d=0

Let $\mathit{RTS}_k(d)$ be the number of 1-side triangles plus 2-side triangles on the right boundary and in direction d. The R,V expansion of figure 7 applies. In the "V" part triangles are swapped $1 \leftrightarrow 2$ sides but their direction is unchanged. The whole of V is turned -1 relative to the desired direction, so the count of d+1 there is required.

$$\mathit{RTS}_k(d) = \mathit{RTS}_{k-1}(d) + \mathit{RTS}_{k-1}(d{+}1)$$

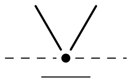
Starting $\mathit{RTS}_0(0) = 1$ and $\mathit{RTS}_0(1) = \mathit{RTS}_0(2) = 0$ gives

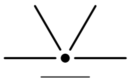
$$\mathit{RTS}_k(d) = \tfrac{1}{3}\ \ 2^k\ + [2,1,-1,-2,-1,1]_{k+2d} \qquad \text{1+2 side triangles by } d$$

$$\begin{aligned}
\mathit{RTS}_k(0) &= 1,1,1,2,5,11,22,43,85,170,341,\ldots && \text{A024493}\\
\mathit{RTS}_k(1) &= 0,0,1,3,6,11,21,42,85,171,342,\ldots && \text{A024495}\\
\mathit{RTS}_k(2) &= 0,1,2,3,5,10,21,43,86,171,341,\ldots && \text{A131708},\ k{\geq}1\ \text{A024494}
\end{aligned}$$

The triangles on the left side of the curve are a 180° rotation. A horizontal d=0 remains horizontal in 180° rotation and similarly d=1 and d=2. So total triangles $2\,\mathit{RTS}_k(d)$.

Count a double-visited point by the direction of its two cross segments. Count a single-visited point by the direction of its absent two cross segments.


$d{=}0$
single-visited


$d{=}0$
double-visited

Let $SD_k(d)$ be the number of single and double points in direction d, excluding the first and last points of the curve which are singles but only one segment at each.

When the curve expands the existing single-visited and double-visited points and their direction are unchanged. Each 1-side or 2-side boundary triangle gives a new single-visited or double-visited point respectively, per theorem 22. A new SD in direction d arises from an RTS triangle direction $d{+}1$.

$$\begin{aligned} SD_k(d) &= \sum_{j=0}^{k-1} RTS_j(d{+}1) \qquad \text{single, double points by } d \\ &= 2\,RTS(k,d)\ -(2 \text{ if } d{=}0) \\ SD_k(0) &= 0,0,0,2,\ \ 8,20,42,84,168,338,\ldots && 2{\times}\text{A111927} \\ SD_k(1) &= 0,0,2,6,12,22,42,84,170,342,\ldots && \text{A086953} \\ SD_k(2) &= 0,2,4,6,10,20,42,86,172,342,\ldots && 2{\times}\text{A131708} \end{aligned}$$

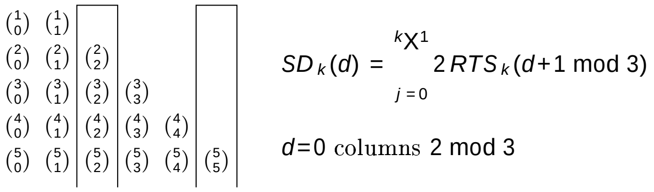
Lines in a given direction have an end at a non-crossing segment of a single or double visited point. For example each SD point $d{=}0$ is the end of a line in directions $d{=}1$ and $d{=}2$. So $Lines(d)$ is SD of directions other than d. The very first and very last points of the curve are ends of a horizontal $d{=}0$.

$$Lines_k(d) = \tfrac{1}{2}\left(SD_k(d{+}1) + SD_k(d{+}2)\right) + (2 \text{ if } d{=}0) \qquad \square$$

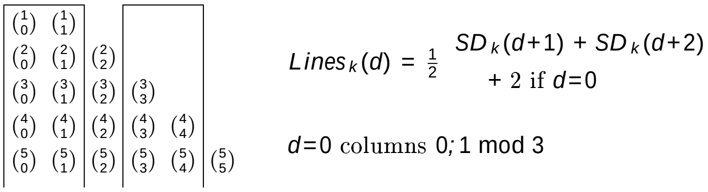
$RTS_k(d)$ is the 3-period binomial sums of Cournot[2], but with a $-d$ meaning $d{=}1$ is the 2 mod 3 binomials and $d{=}2$ is the 1 mod 3 binomials.

$$RTS_k(d) = \binom{k}{-d} + \binom{k}{-d+3} + \binom{k}{-d+6} + \cdots \qquad d=0,1,2$$

The sum in $SD_k(d)$ is total of those binomials in columns down to row $k{-}1$.



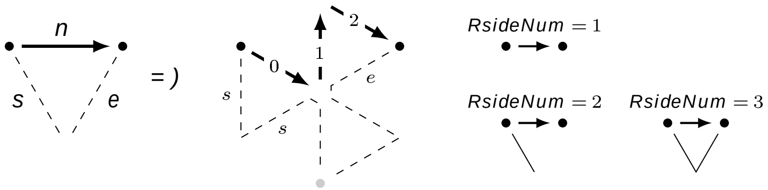
Then $Lines_k(d)$ is the "other" two $SD_k(d)$ which means 2 out of 3 columns to row k.



RTS_k combines 1-side and 2-side triangles, and SD_k combines 1 and 2 points, since those combinations suffice for the lines calculation. The 1s and 2s can be counted separately if desired and they are mod 6 columns of the binomials. When expressed as powers they have a 12-period half-power term $3^{\lfloor k/2 \rfloor}$. By taking 1s and 2s together those half-powers cancel out leaving just a 6-period constant term.

6 Enclosure Sequence

When a segment is appended to the curve it can be the first, second or third segment of the unit triangle on its right. Let $\mathit{RsideNum}(n) = 1, 2, 3$ be the side number of n on that triangle. A segment may have one or both segments s or e as follows,



The expansion shows how a segment with s and/or e expands to a new combination. For new low digit 1 on n it can be noted that segment 2 is after n so is not yet present. This means e occurs only with s so there is only a single $\mathit{RsideNum} = 2$ form.

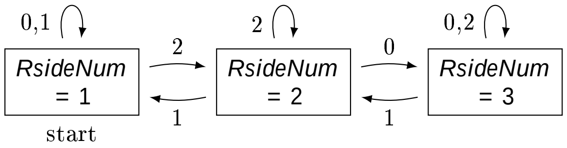
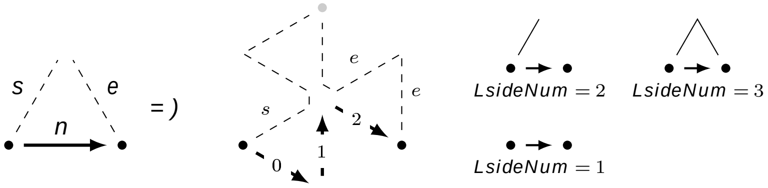


Figure 12: n ternary high to low

$$\begin{aligned} \mathit{RsideNum}(n) &= \text{figure 12 final state} \\ &= 1, 1, 2, 1, 1, 2, 3, 1, 2, 1, 1, 2, 1, 1, 2, 3, 1, 2, \ldots \end{aligned}$$

$$\begin{aligned} &= 1 \text{ at } \quad n = 0, 1, 3, 4, 7, 9, 10, 12, 13, 16, \ldots \\ &= 2 \text{ at } \quad n = 2, 5, 8, 11, 14, 17, 19, 23, 26, 29, \ldots \\ &= 3 \text{ at } \quad n = 6, 15, 18, 20, 24, 33, 42, 45, 47, 51, \ldots \end{aligned}$$

Left side segments follow similarly



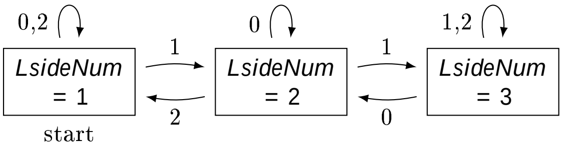


Figure 13: ternary high to low

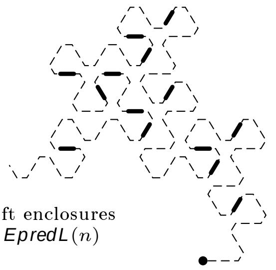
$$\begin{aligned}
\textit{LsideNum}(n) &= \text{figure 13 final state}\\
&\qquad = 1,2,1,2,3,1,1,2,1,2,3,1,2,3,3,1,2,1,\dots\\
= 1 \text{ at} \quad & n = 0,2,5,6,8,11,15,17,18,20,\dots\\
= 2 \text{ at} \quad & n = 1,3,7,9,12,16,19,21,25,27,\dots\\
= 3 \text{ at} \quad & n = 4,10,13,14,22,28,31,32,37,40,\dots
\end{aligned}$$

LsideNum state machine figure 13 is a reversal of *RsideNum* state machine figure 12. Digits are reversed 0↔2 and the side number reversed 1↔3.

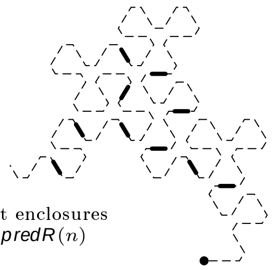
Geometrically this is simply the curve being the same in 180° rotation, so that the left side counted from the end is the same as the right side counted forward. The reversal of the side number counts downwards from how many sides it will have, so

$$\textit{LsideNum}(3^k{-}1-n) \;=\; \textit{Rsides}(n){+}1-\textit{RsideNum}(n)$$

$\textit{RsideNum}(n) = 3$ is where n encloses a unit triangle on the right. Similarly $\textit{LsideNum}(n) = 3$ on the left.



left enclosures
$\textit{EpredL}(n)$



right enclosures
$\textit{EpredR}(n)$

$$\begin{aligned}
\textit{EpredR}(n) &= \begin{cases} 1 & \text{if } \textit{RsideNum}(n) = 3\\ 0 & \text{if not}\end{cases}\\
&= \begin{cases} 1 & \text{if pair 20 and any 1s below it are in pairs 10}\\ 0 & \text{otherwise}\end{cases} \qquad (56)\\
&= 0,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,0,\ 0,0,0,\ 1,0,0,\ \dots\\
\textit{EpredL}(n) &= \begin{cases} 1 & \text{if } \textit{LsideNum}(n) = 3\\ 0 & \text{if not}\end{cases}\\
&= 0,0,0,\ 0,1,0,\ 0,0,0,\ 0,1,0,\ 0,1,1,\ 0,0,0,\ \dots
\end{aligned}$$

Form (56) is since 20 in figure 12 goes to or stays in side 3. A 1 digit would leave there, unless it's a 10 pair which goes back. The digit 2 loop in side 2 would be another 20 if it goes back to side 3 that way.

Some usual state machine manipulations can take digits of n low to high instead. Reaching *EpredR* is a right enclosure. Reaching "not" or ending in rm0 or rm1 is not. Similarly *EpredL*.

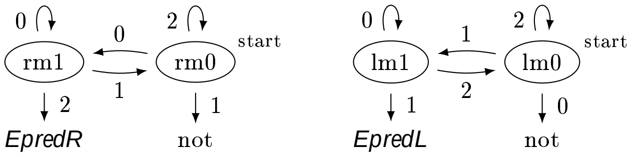
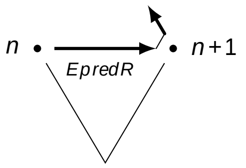


Figure 14:
ternary
low to high

Each enclosure is an enclosed unit triangle on the respective side right or left, so totals AR and AL from theorem 18.

$$\mathit{AR}_k \;=\; \mathit{AL}_k \;=\; \sum_{n=0}^{3^k-1} \mathit{EpredR}(n) \;=\; \sum_{n=0}^{3^k-1} \mathit{EpredL}(n) \tag{57}$$

When $\mathit{EpredR}(n)$ encloses a unit triangle the next turn is left $\mathit{turn}(n{+}1) = +1$, since otherwise the next segment would overlap the triangle just enclosed. Conversely EpredL is followed by a right turn



$\mathit{turn}(n{+}1) = 1$ left
or would overlap segment
of triangle just enclosed

As from subsection 1.2, a left turn at $n{+}1$ is $\mathit{LowestNonTwo}(n) = 0$. For EpredR in figure 14, low 2s loop in rm0 and then if a 1 go to "not" so never a right turn. For EpredL conversely 0 goes to "not" so never left turn.

EpredR can enclosure 2 triangles consecutively. This occurs first at $n{=}56, 57$ which are ternary 2002 and 2010. There cannot be 3 or more consecutive EpredR or that would be 3 left turns and the segments would overlap. Similarly EpredL pair, which first occurs at $n{=}13, 14$, ternary $111, 112$.

Some state machine manipulations can test whether $n{+}1$ is also the respective enclosure, then intersection n and $n{+}1$ for a pair. Taking that low to high shows enclosure pairs are the original digit forms with extra low.

$$\mathit{EpredRpair} = \begin{array}{|c|c|c|}\hline \mathit{EpredR} & 0 & 2\,\dots\,2 \\ \hline \end{array} \qquad \mathit{EpredLpair} = \begin{array}{|c|c|}\hline \mathit{EpredL} & 1 \\ \hline \end{array}$$

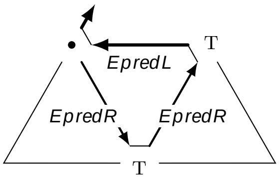
$$\underbrace{}_{\geq 1 \text{ digits}}$$

The last segment of curve k is not an enclosure, since it is the first visit to its endpoint, so pairs do not cross a level. The number of pairs within a level follow from (57) and the extra digits.

$$\begin{aligned} \sum_{n=0}^{3^k-1} \mathit{EpredRpair}(n) &= \sum_{h=0}^{k-2} \mathit{AR}_h \;=\; \begin{cases} 0 & \text{if } k{=}0 \\ \tfrac{1}{2}T_{k-1} & \text{if } k \geq 1 \end{cases} \qquad (58) \\ &= 0, 0, 0, 0, 1, 6, 25, 90, 301, 966, \ldots \qquad \text{A000392} \\ \sum_{n=0}^{3^k-1} \mathit{EpredLpair}(n) &= \mathit{AL}_k \end{aligned}$$

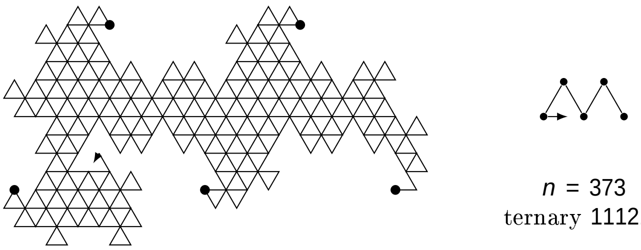
At (58), cumulative AR is $\frac{1}{2}T$ in the manner of theorem 22. New triples are formed when segments expand into each triangle A, here it is just AR triangles so half. The result is the Stirling numbers of the second kind.

When 2 consecutive EpredR occur the next segment is always an EpredL left enclosure, since it was 2 left turns. Conversely 2 consecutive EpredL is always followed by EpredR.



2 right enclosures
are 2 left turns T
so next segment
is left enclosure

Runs of right and left enclosures can occur. For example at n=373 ternary 111211 there is a run of 12 consecutive enclosures. The following diagrams show how this run falls within its surrounding segments.



n = 373
ternary 111211

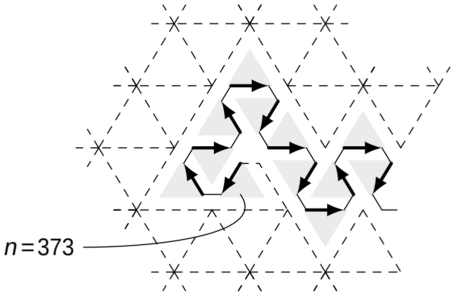


Figure 15:
enclosure sides
LLR, LLR,
LRR, LLR

There are no runs longer than 12. That can be seen by some state machine manipulations on *Epred* left or right to ask whether n+1, n+2 etc also enclosing. The intersection of *Epred* on 13 terms n through n+12 inclusive is empty.

State machine manipulations on the 12 intersection shows it is *EpredL* with some extra low digits,

$$\mathit{EpredTwelve} = \boxed{\mathit{EpredL}}\boxed{1}\underbrace{\boxed{2{:}{:}{:}2}}_{\geq 1\text{ digits}}\boxed{11} \qquad \text{ternary} \tag{59}$$

The count of how many 12 runs in k is the same as *EpredRpair* in k−2. The digit form for *EpredTwelve* is like *EpredRpair* but with 2 extra fixed digits. The high is *EpredL* rather than *EpredR*, but their counts are the same (57).

Runs of 12 all have the same enclosure side sequence shown in figure 15. This can be seen from *turn*(n+1) which is opposite to the enclosed side. It is *LowestNonTwo* on low digits of 1211 through 2020 of *EpredTwelve* at (59), and is the same when more 2s for 12…211 there.

Each n is visit number 1, 2 or 3 to its point. This is given by *RsideNum* or *LsideNum* when the sides of such a triangle expand to meet in the middle.

$n \equiv 1 \bmod 3$ is the right side or $n \equiv 2 \bmod 3$ is the left side, and then any number of low 0s since those 0s do not change existing points.

$$
\mathit{VisitNum}(n) = \begin{cases} 1 & \text{if } n{=}0 \\ \mathit{RsideNum}(m) & \text{if } n = (3m{+}1).3^l \\ \mathit{LsideNum}(m) & \text{if } n = (3m{+}2).3^l \end{cases}
$$
$$
= 1, 1, 1, 1, 1, 2, 1, 2, 1, 1, 1, 2, 1, 1, 3, 2, 2, 1, 1, 3, \ldots
$$

$$
\boxed{\mathit{RsideNum}}\boxed{1}\underbrace{\boxed{0 \ldots 0}}_{\geq 0 \text{ digits}} \quad \text{or} \quad \boxed{\mathit{LsideNum}}\boxed{2}\underbrace{\boxed{0 \ldots 0}}_{\geq 0 \text{ digits}}
$$

The visit number is also how many $\mathit{other}(n, \delta)$ are on the same arm and preceding n.

$$
\mathit{VisitNum}(n) = 1 + \operatorname*{count}_{\delta=1,2} \ \mathit{other}(n, \delta) \text{ same arm and } < n
$$

or count with $\delta{=}0$ to include n itself unchanged

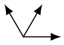
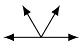
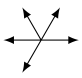
$$
\mathit{VisitNum}(n) = \operatorname*{count}_{\delta=0,1,2} \ \mathit{other}(n, \delta) \text{ same arm and } \leq n
$$

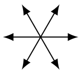
Total of *VisitNum* within level k counts 1 each single, 1+2 each double, and 1+2+3 each triple,

$$
\begin{aligned} \sum_{n=0}^{3^k} \mathit{VisitNum}(n) = S_k + 3D_k + 6T_k \quad &= 2.3^k - 2.2^k + 2 \\ &= 2, 4, 12, 40, 132, 424, 1332, \ldots \qquad 2{\times}\text{A083323} \end{aligned}
$$

7 Multiple Arms

Six copies of the terdragon at 60° angles mesh perfectly and fill the plane (theorem 2). The boundary of 2 to 6 such arms can be calculated simply as R_k (39) on the ends and one or more V_k (40) in between. The area follows from the boundary by (47).

Arms		Boundary	Area
2		$\begin{cases} 4 \\ 9.2^{k-1} \end{cases}$	$\begin{cases} 0 & \text{if } k = 0 \\ 4.3^{k-1} - 3.2^{k-1} & \text{if } k \geq 1 \end{cases}$
3		12.2^{k-1}	$6.3^{k-1} - 4.2^{k-1}$
4		$\begin{cases} 8 \\ 15.2^{k-1} \end{cases}$	$\begin{cases} 0 & \text{if } k = 0 \\ 8.3^{k-1} - 5.2^{k-1} & \text{if } k \geq 1 \end{cases}$
5		$\begin{cases} 10 \\ 18.2^{k-1} \end{cases}$	$\begin{cases} 0 & \text{if } k = 0 \\ 10.3^{k-1} - 6.2^{k-1} & \text{if } k \geq 1 \end{cases}$

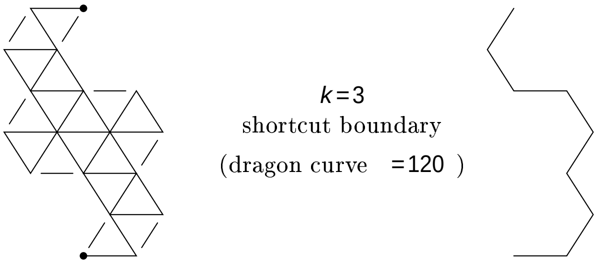
6		$\begin{cases} 12 \\ 18.2^{k-1} \end{cases}$	$\begin{cases} 0 & \text{if } k=0 \\ 12.3^{k-1} - 6.2^{k-1} & \text{if } k \geq 1 \end{cases}$

The boundary increases by an extra V_k with each extra arm. For 3 arms the k=0 and $k \geq 1$ cases coincide.

In 5 arms the gap is $2R_k$ and in 6 arms the corresponding section is $2V_k$. With $R_k = V_k$ for $k \geq 1$ from (39)(40) the 5 and 6 arm curves are $B6(k) = B5(k)$ for $k \geq 1$.

8 Shortcut Boundary

The terdragon boundary has "V" notches at every third boundary position. These are the 2-side boundary triangles $\mathit{BT2}_k$ from theorem 10 and the -1 boundary turns from theorem 17. A variation on the curve can be made by taking shortcuts across those Vs.



Theorem 25.

$$\mathit{BSH}_k = 2^{k+1}$$
$$\mathit{RSH}_k = \mathit{BSH}_k/2 = 2^k$$

$$\mathit{ASH}_k = \begin{cases} 0 & k=0 \\ 2.3^{k-1} & k \geq 1 \end{cases} \tag{60}$$

The shortcuts add the 2-sided boundary triangles as additional area,

$$\begin{aligned} \mathit{ASH}_k &= A_k + \mathit{BT2}_k \\ &= \begin{cases} 0+0 & \text{if } k=0 \\ 2(3^{k-1} - 2^{k-1}) + 2^k & \text{if } k \geq 1 \end{cases} \end{aligned}$$

The shortcuts shorten the boundary by 1 side at each 2-sided boundary triangle,

$$\begin{aligned} \mathit{BSH}_k &= B_k - \mathit{BT2}_k \\ &= \begin{cases} 2+0 & \text{if } k=0 \\ 3.2^k - 2^k & \text{if } k \geq 1 \end{cases} \end{aligned}$$

□

The shortcuts maintain the three-sides-enclosed property of lemma 1 and so shortcut area and boundary are related to total line segments by

$$3\,\mathit{ASH}_k + \mathit{BSH}_k = 2\,(3^k + \mathit{BT2}_k)$$

Riddle[8] takes this shortcut curve form to show the terdragon as a fractal has area $1/(2\sqrt{3})$. Scaling ASH_k by the curve endpoint distance $\sqrt{3}^{\,k}$ squared gives

$$\frac{\mathit{ASH}_k}{(\sqrt{3})^{2k}} = \frac{2.3^{k-1}}{3^k} = \frac{2}{3} \qquad \text{of base triangle area}$$

A base equilateral triangle of unit side has height $\frac{1}{2}\sqrt{3}$ so area $\frac{1}{4}\sqrt{3}$, giving

$$\frac{2}{3} \cdot \tfrac{1}{4}\sqrt{3} = \frac{1}{2\sqrt{3}} = 0.288675\ldots \qquad \text{A020769}$$

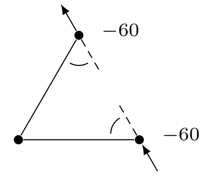
Going instead from the plain enclosed area A_k (48) the result is the same

$$\frac{\frac{\sqrt{3}}{4} A_k}{(\sqrt{3})^{2k}} = \frac{1}{2\sqrt{3}} - \tfrac{\sqrt{3}}{4}\ \tfrac{2}{3}\ ^{k} \to \frac{1}{2\sqrt{3}}$$

Theorem 26.

$$\theta = 120°$$

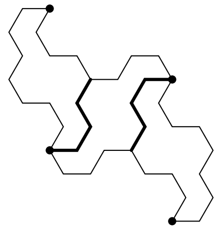
In turn sequence $\mathit{Rt}(i)$ from theorem 17 the -1 turns are eliminated leaving just the dragon turns. The turns before and after the shortcut are both reduced by 60°. In $\mathit{Rt}(i)$ the turns $+120°$ and 0 become $+60°$ and $-60°$ respectively. Those 60° turns correspond to unfolding the dragon by $\theta = 120°$.



turns before and
after shortcut
reduced by 60

□

The shortcut area (60) has $\mathit{ASH}_{k+1} = 3\,\mathit{ASH}_k$ for $k \geq 1$ so the area is exactly 3 copies of the previous level, with no join area in between.



k=3
shortcut boundary

join length
$\mathit{JBSH}_3 = 4$

Theorem 27.

$$\mathit{JBSH}_k = 2^{k-1} \qquad k \geq 1$$

For $k \geq 1$ the total shortcut boundary BSH_{k+1} is 3 copies of the previous level boundary less 4 copies of the join boundary (2 in each join).

$$\begin{aligned} \mathit{BSH}_{k+1} &= 3\,\mathit{BSH}_k - 4\,\mathit{JBSH}_k \\ \mathit{JBSH}_k &= (3.2^{k+1} - 2^{k+2})/4 \ = \ 2^{k-1} \end{aligned} \qquad \square$$

Exact matching of the shortcut sides can also be seen in the dragon curve turn sequence of theorem 26. In a dragon curve with 2^k segments the turns in the second half are reverse order and opposite direction to the first half, so the second half of one boundary matches the first half of the next. (It would then have to be shown that the matching goes no further.)

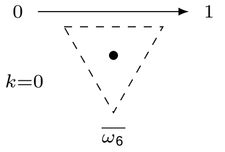
9 Centroid

The terdragon curve is symmetric in 180° rotation so the centroid of the segments, points or area are all the midpoint of the curve at $b^k/2$. But some measures can be made on just one side of the curve.

Theorem 28. k

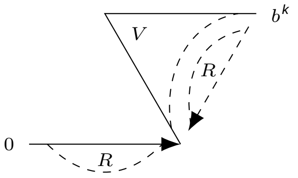
$$\begin{aligned} \mathit{GRT}_k &= \frac{7-2\omega_6}{13}\, b^k \ + \ \frac{5-7\omega_6}{39} \ \frac{\overline{\omega_6}}{2}^{\;k} \\ &= \tfrac{3-\sqrt{3}i}{6},\ \tfrac{9+\sqrt{3}i}{12},\ \tfrac{24+14\sqrt{3}i}{24},\ \tfrac{33+67\sqrt{3}i}{48},\ \tfrac{-99+233\sqrt{3}i}{96},\ \ldots \end{aligned}$$

For k=0 the curve is a single line segment with a single triangle. The centroid of the triangle is the mean of its corners.



$$\mathit{GRT}_0 = \frac{0+1+\overline{\omega_6}}{3} = \frac{\overline{b}}{3}$$

As in theorem 9, the boundary triangles in a V part are a reversal of the R part, so the centroid is the mean of the two copies in the previous level.



$$\begin{aligned} \mathit{GRT}_k &= \tfrac{1}{2}\ \mathit{GRT}_{k-1} \ + \ b^k + (\omega_6)^4 \mathit{GRT}_{k-1} \\ &= \tfrac{\overline{\omega_6}}{2} \mathit{GRT}_{k-1} \ + \ \tfrac{1}{2} b^k \\ &= \mathit{GRT}_0 \ \tfrac{\overline{\omega_6}}{2}^{\;k} \ + \ \tfrac{1}{2} b \sum_{j=0}^{k-1} \ \tfrac{\overline{\omega_6}}{2}^{\;j}\, b^{k-1-j} \end{aligned}$$

$$= \frac{\bar{b}}{3}\ \frac{\overline{\omega_6}}{2}\ ^{k} + \tfrac{1}{2} b \frac{\frac{\overline{\omega_6}}{2}\ ^{k} - b^k}{\frac{\overline{\omega_6}}{2}\ - b} \qquad \square$$

Per theorem 26, the line segments of the shortcut boundary are the Heighway/Harter dragon curve unfolding by 120°. The same reversing calculation as above is made for its centroid, but with initial line centroid $\mathit{GRSH}_0 = \frac{1}{2}$. Equating the sum parts of the two gives

$$\mathit{GRSH}_k - \mathit{GRSH}_0 .\ \frac{\overline{\omega_6}}{2}\ ^{k} = \mathit{GRT}_k\ -\ \mathit{GRT}_0 .\ \frac{\overline{\omega_6}}{2}\ ^{k}$$

$$\mathit{GRSH}_k = \frac{7-2\omega_6}{13} b^k + \frac{-1+4\omega_6}{26}\ \frac{\overline{\omega_6}}{2}\ ^{k} \qquad \text{terdragon } 120^\circ \text{ centroid}$$

$$= \tfrac{2}{4}, \tfrac{7+\sqrt{3}i}{8}, \tfrac{17+9\sqrt{3}i}{16}, \tfrac{22+44\sqrt{3}i}{32}, \tfrac{-67+155\sqrt{3}i}{64}, \ldots$$

Theorem 29. k

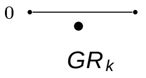
$$\mathit{GR}_k = \begin{cases} \frac{1}{2} & k = 0 \\ \mathit{GRT}_k + \frac{\omega_6}{3}\ \frac{b}{2}\ ^{k} & k \geq 1 \end{cases}$$

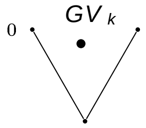
$$= \tfrac{2}{4}, \tfrac{9+3\sqrt{3}i}{12}, \tfrac{21+17\sqrt{3}i}{24}, \tfrac{24+70\sqrt{3}i}{48}, \tfrac{-117+233\sqrt{3}i}{96}, \ldots$$

$$\mathit{GV}_k = \begin{cases} \frac{1}{2} - \frac{1}{4}\sqrt{3}i & k = 0 \\ \mathit{GRT}_k - \frac{\omega_6}{3}\ \frac{b}{2}\ ^{k} & k \geq 1 \end{cases}$$

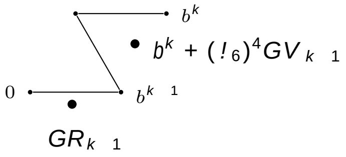
$$= \tfrac{4-2\sqrt{3}i}{8}, \tfrac{9-\sqrt{3}i}{12}, \tfrac{27+11\sqrt{3}i}{24}, \tfrac{42+64\sqrt{3}i}{48}, \tfrac{-81+233\sqrt{3}i}{96}, \ldots$$

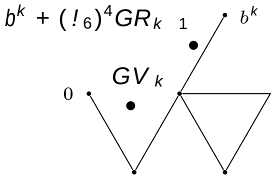
The centroid of the R right and V part boundaries are





These parts expand, similar to the R,V expansion of figure 8,





For $k \geq 1$ there are the same number of segments $R_k = V_k$ in each part so the centroids are the mean of the previous level.

$$\mathit{GR}_k = \tfrac{1}{2}\mathit{GR}_{k-1}\ +\ \tfrac{1}{2}\ \ b^k + (\omega_6)^4 \mathit{GV}_{k-1} \qquad k \geq 2 \tag{61}$$

$$\mathit{GV}_k = \tfrac{1}{2}\mathit{GV}_{k-1}\ +\ \tfrac{1}{2}\ \ b^k + (\omega_6)^4 \mathit{GR}_{k-1} \tag{62}$$

Taking (61) for GV and substituting into (62) gives

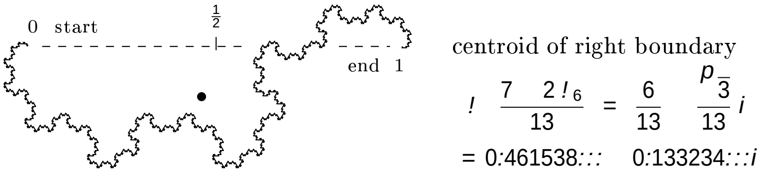
$$GR_k = GR_{k-1} - \tfrac{\bar{b}}{4} GR_{k-2} + \tfrac{1}{4} b^k \qquad k \geq 3$$

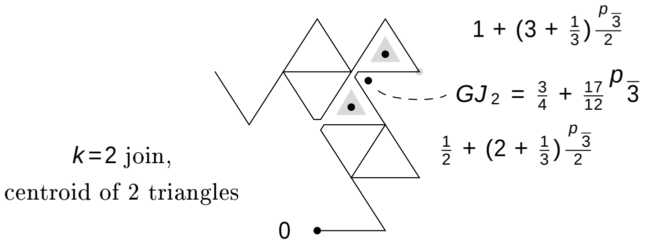
The characteristic polynomial of the GR terms alone is

$$x^2 - x + \tfrac{\bar{b}}{4} = (x - \tfrac{\overline{\omega_6}}{2})(x - \tfrac{b}{2})$$

so GR_k is powers of $\frac{\overline{\omega_6}}{2}$, $\frac{b}{2}$ and the further b. From the initial values the coefficients of b and $\frac{\overline{\omega_6}}{2}$ are the same as for GRT_k. The coefficient of the $\frac{b}{2}$ power is $\frac{\omega_6}{3}$. Substituting into (61) gives GV_k in the same form but coefficient $-\frac{\omega_6}{3}$. ☐

For the terdragon fractal, all four right boundary centroid forms above can be scaled by b^k for a unit length curve. The limit as $k \to \infty$ is the coefficient of the b^k term and so is the same in each case. Notice this is not the middle horizontally but a little towards the start at $\frac{6}{13}$

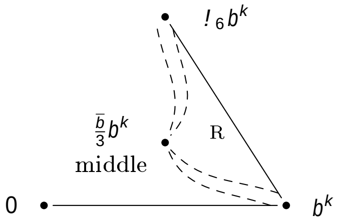




Theorem 30. $k \geq 1$
k

$$\begin{aligned} GJ_k &= b^{k+1} - 2\omega_6 GRT_k & k \geq 1 \\ &= \frac{9+3\omega_6}{13} b^k + \frac{-14+4\omega_6}{39} \; \frac{\overline{\omega_6}}{2}^{\;k} & (63) \\ &= 1+\tfrac{2}{3}\sqrt{3}i,\ \tfrac{3}{4}+\tfrac{17}{12}\sqrt{3}i,\ -1+\tfrac{29}{12}\sqrt{3}i,\ -\tfrac{83}{16}+\tfrac{149}{48}\sqrt{3}i,\ \ldots \end{aligned}$$

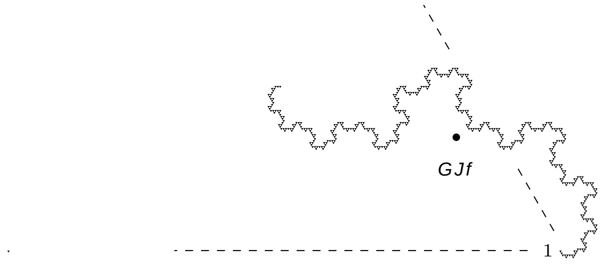
For $k \geq 1$ the right boundary triangles are two joins, per the triangle arrangement in the second proof of area theorem 18. So, with suitable rotations and offsets, the mean of the join centroids is the right triangles centroid GRT_k.



$$\begin{aligned} \tfrac{1}{2}\mathit{GJ}_k + \tfrac{1}{2}\ \ b^k + (\omega_6)^2 \mathit{GJ}_k &= \omega_6 b^k + (\omega_6)^5 \mathit{GRT}_k \\ \mathit{GJ}_k = \frac{\omega_6 b^k + (\omega_6)^5 \mathit{GRT}_k - \frac{1}{2}b^k}{\frac{1}{2} + \frac{1}{2}(\omega_6)^2} & \end{aligned} \qquad \square$$

Scaled by b^k for a fractal of unit length, the limit is the coefficient of the b^k term in (63).

$$\frac{\mathit{GJ}_k}{b^k} \to \mathit{GJf} = \frac{9 + 3\omega_6}{13} = \frac{21 + 3\sqrt{3}i}{26} = 0.807692\ldots + 0.199852\ldots i$$



Theorem 31.

$$k \geq 2$$

$$\begin{aligned} \mathit{GAR}_k &= \frac{1}{2}b^k \ + \ \frac{1}{156}.\frac{(-3{+}12\omega_6)2^k b^k \ - \ 26\omega_6 b^k \ + \ (-10{+}14\omega_6)\,\overline{\omega_6}^k}{3^{k-1} - 2^{k-1}} \\ &= \tfrac{3+5\sqrt{3}i}{6}, \tfrac{-12+46\sqrt{3}i}{30}, \tfrac{-306+248\sqrt{3}i}{114}, \tfrac{-2769+799\sqrt{3}i}{390}, \ldots \qquad k \geq 2 \end{aligned}$$

Each segment is either a right boundary or a side of a right-side enclosed unit triangle. Weighted by the number of segments, the centroid of the enclosed triangles and the boundary segments sum to the centroid of all segments which is the midpoint $\frac{1}{2}b^k$.

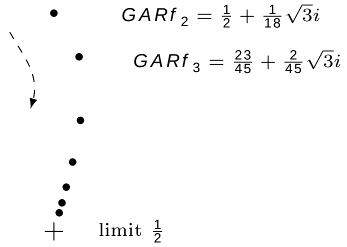
$$3^k.\tfrac{1}{2}b^k \ = \ 3\mathit{AR}_k.\mathit{GAR}_k \ + \ R_k.\mathit{GR}_k \qquad \square$$

The right side area is three copies of the previous level and one join, so $\mathit{AR}_k = 3\mathit{AR}_{k-1} + J_{k-1}$. The centroids of those give a recurrence for GAR with the join centroid GJ.

$$GAR_k = \frac{1}{AR_k}\left(\begin{array}{l} AR_{k-1}.GAR_{k-1} \\ +AR_{k-1}.\ \ b^{k-1}+\omega_3 GAR_{k-1} \\ +AR_{k-1}.\ \ \omega_6 b^{k-1}+GAR_{k-1} \\ +\ \ J_{k-1}.\ \ b^{k}+GJ_{k-1} \end{array}\right)$$

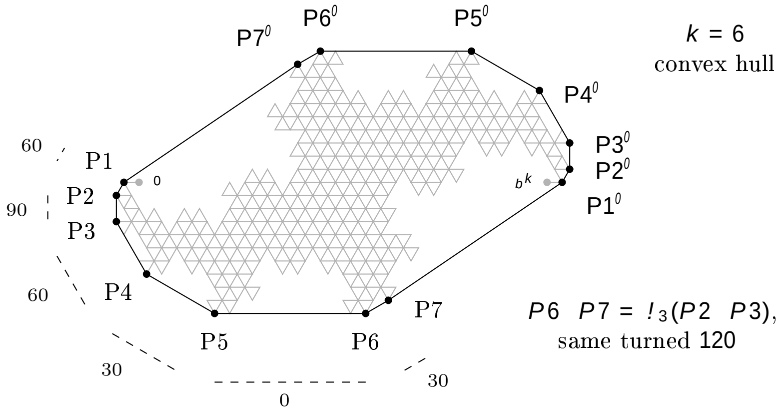
Scaled by b^k to make a fractal of unit length the limit is $\frac{1}{2}$ which is the midpoint of the whole.

$$GARf_k = GAR_k/b^k \to \tfrac{1}{2} \quad \text{as } k \to \infty$$



10 Convex Hull

A convex hull is the smallest convex polygon which can be drawn around a given set of points.



Theorem 32. $k \geq 6$

$$\begin{aligned} P1(k) &= -\tfrac{1}{24}\ \ b^{k} + p(k) \\ P2(k) &= -\tfrac{1}{24}\ \ b^{k+1} + p(k{+}1) \\ P3(k) &= -\tfrac{1}{24}\ \ b^{k+2} + p(k{+}2) \\ P4(k) &= -\tfrac{1}{24}\ \ b^{k+3} + p(k{+}3) \\ P5(k) &= -\tfrac{1}{24}\ \ b^{k+4} + p(k{+}4) \\ P6(k) &= -\tfrac{1}{24}\ \ b^{k+5} + p(k{+}5) \\ P7(k) &= -\tfrac{1}{24}\ \ (1{+}\tfrac{1}{9}\omega_6)b^{k+5} + p(k{+}6) \end{aligned} \tag{64}$$

$$
\begin{array}{llll}
p(m) = [-9, & 6+15\omega_3, & -3-3\omega_3, & -3-6\omega_3, \\
\quad -9\omega_3, & (6+15\omega_3)\omega_3, & (-3-3\omega_3)\omega_3, & (-3-6\omega_3)\omega_3, \\
\quad -9\omega_3^2, & (6+15\omega_3)\omega_3^2, & (-3-3\omega_3)\omega_3^2, & (-3-6\omega_3)\omega_3^2\,] \\
\quad m \equiv 0 \text{ to } 11 \bmod 12
\end{array}
$$

$$
\begin{array}{lll}
P1'(k) = b^k - P1(k), & P4'(k) = b^k - P4(k), & P6'(k) = b^k - P6(k), \\
P2'(k) = b^k - P2(k), & P5'(k) = b^k - P5(k), & P7'(k) = b^k - P7(k) \\
P3'(k) = b^k - P3(k), & &
\end{array}
$$

$+30°$

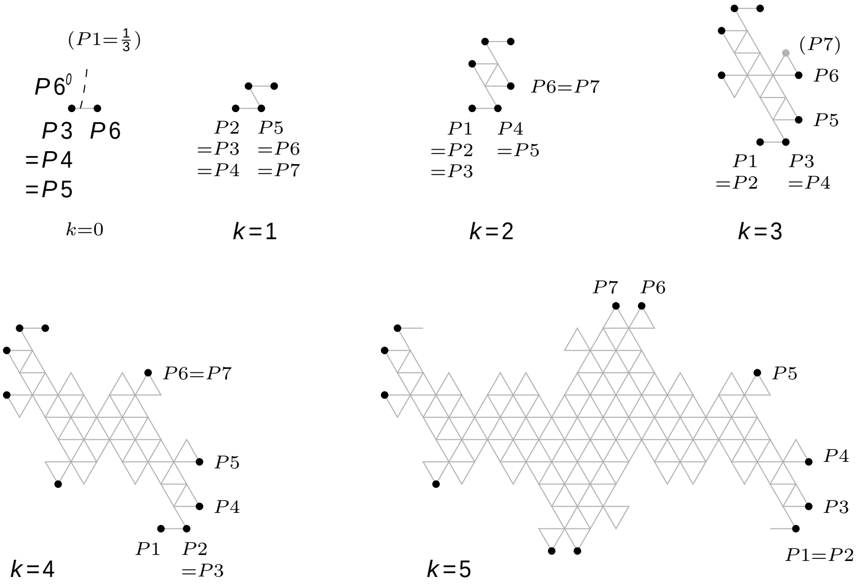
$+120°$ $P1'$

$k < 6$

k

$P3{=}P4{=}P5{=}P6'$ $P1$ $P2, P7$
$P2{=}P3{=}P4$ $P5{=}P6{=}P7$ $P1$
$P1{=}P2{=}P3$ $P4{=}P5$ $P6{=}P7$
$P1{=}P2$ $P3{=}P4$ $P7$
$P2{=}P3$ $P6{=}P7$
$P1{=}P2$

For $k = 0$ to 5 the convex hulls can be formed explicitly. For $k{=}0$ the hull is merely a line 0 to 1. $P1 = \frac{1}{3}$ is on that line but not a vertex. For $k{=}3$ point $P7 = P6 + \omega_3$ is on the boundary but not a vertex.



Side P1–P2 is at 60° relative to the b^k endpoint since

$$\frac{P1(k) - P2(k)}{b^{k+2}} = \tfrac{1}{72} + \tfrac{1}{24}\frac{p(k+1) - p(k)}{b^{k+2}}$$

and the periodic values of $p(m)$ have difference $p(k+1) - p(k)$ which is always aligned to the b^{k+2} direction. These p differences can be illustrated

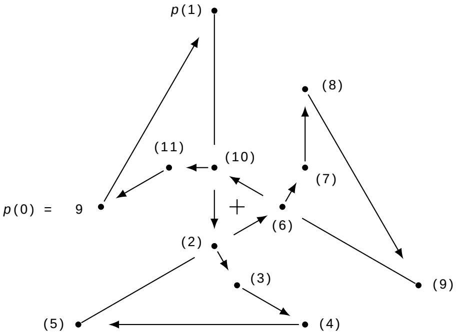


Figure 16: $p(m)$ steps

$p(0)$ to $p(1)$ at the top left is 60° since $p(1) - p(0) = 15\omega_6$ corresponding to b^2. At each point the direction turns +30° the same as $\arg b = 30°$. At $m = 0, 1, 4, 6, 8, 9$ there is an additional reversal 180° but still +30°.

Similarly the other sides P2–P3 aligned to b^{k+3} etc through P6–P7 aligned to b^{k+7}.

The sides P2–P3 and P6–P7 are the same length but turned +120° since, using $b^4 = 9\omega_3$ and $p(m+4) = \omega_3 p(m)$,

$$P2(k) - P3(k) \;=\; \omega_3 \;\; P6(k) - P7(k)$$

For the vertex formulas, proceed by induction. Suppose the formulas are true of $k-1$. Terdragon k comprises three $k-1$. The convex hull around k is the hull around the hulls of the three sub-parts.

The expansion is shown in the following diagram. 0 is the origin. b^k is the endpoint of level k. The three sub-parts are A,B,C and their vertices are labelled $P1A$, $P1B$, $P1C$ etc.

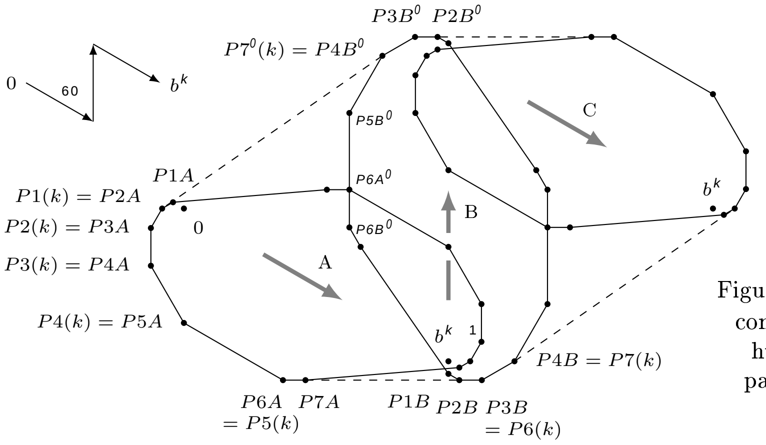


Figure 17: convex hull parts

For the dashed bottom side, both P6A–P7A and P2B–P3B are horizontal (aligned to the b^k endpoint) as per the side angles above and the respective A

and B parts turned $-30°$ and $+90°$. They are at the same position vertically since, with $p(k{+}5) - p(k{+}4)$ aligned to b^k (the bottom horizontal $p(4)$ to $p(5)$ in figure 16),

$$\operatorname{Im} \frac{P6A - P3B}{b^k} = \operatorname{Im} \quad -\tfrac{3}{8} + \tfrac{1}{24} \frac{p(k{+}5) - p(k{+}4)}{b^k} \qquad = 0$$

So the hull is $P5(k)$ at P6A across to $P6(k)$ at P3B.

For the dashed top left P2A–P4B′, the sub-part sides P1A–P2A and P3B′–P4B′ are both 60° per the side angles. But P2A–P4B′ is steeper than 60° since

$$\begin{aligned}\operatorname{Im} \frac{P2A - P4B'}{b^{k+2}} &= \operatorname{Im} \quad -\tfrac{1}{9} + \tfrac{1}{12}\omega_3 - \tfrac{1}{72} \frac{p(k{+}10) + p(k{+}4)}{b^k} \\ &= \tfrac{1}{24}\sqrt{3} \quad 1 - (-\tfrac{1}{3})^{\lceil k/2 \rceil} \qquad > 0 \qquad \text{for } k \geq 7\end{aligned}$$

So P1A is inside the hull and $P1(k)$ is at P2A. Likewise at the top $P7'(k)$ is at P4B′. The side P1A–P2A is quite short so a little difficult to see in figure 17.

The other new sides are the same rotated 180°.

So mutual recurrences for the vertices

$$\begin{aligned} P1(k) &= P2(k{-}1) & \qquad P5(k) &= P6(k{-}1) \\ P2(k) &= P3(k{-}1) & P6(k) &= b^{k-1} + \omega_3 P3(k{-}1) \\ P3(k) &= P4(k{-}1) & P7(k) &= b^{k-1} + \omega_3 P4(k{-}1) \\ P4(k) &= P5(k{-}1) \end{aligned}$$

The power forms (64) of the theorem satisfy these recurrences starting from an initial $k{=}6$ hull calculated explicitly, which completes the induction. The power forms can be found by writing the recurrences in generating functions and solving simultaneously with some linear algebra, or directly by expanding. The chain of dependencies is

$$\begin{tikzcd} \mathrm{P1} \arrow[r] & \mathrm{P2} \arrow[r] & \mathrm{P3} \arrow[r] & \mathrm{P4} \arrow[d] & \mathrm{P7} \arrow[l] \\ & & \mathrm{P6} \arrow[u] & \mathrm{P5} \arrow[l] & \end{tikzcd}$$

Starting at $P3(k)$ and expanding to reach $P3(k{-}4)$ again,

$$P3(k) = b^{k-4} + \omega_3 P3(k{-}4)$$

Apply this repeatedly until reaching $k = 6, 7, 8$ or 9. Let this be $q \geq 0$ many times so that $k{-}6 = 4q + r$ with $0 \leq r \leq 3$ so ending at $P1(6{+}r)$.

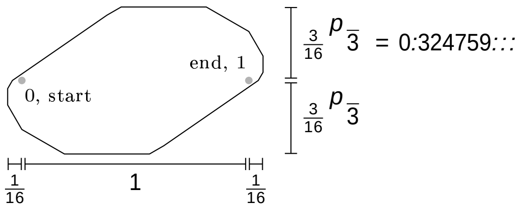
$$\begin{aligned} P3(k) &= b^{k-4} + \omega_3 b^{k-8} \cdots + {\omega_3}^{q-1} b^{k-4-4(q-1)} + {\omega_3}^q P3(6{+}r) \\ &= {\omega_3}^q b^{r+6} \frac{(b^4)^q - {\omega_3}^q}{b^4 - \omega_3} \quad + {\omega_3}^q P3(6{+}r) \\ &= -\tfrac{1}{24} \quad b^{k+2} \qquad - b^{r+8}{\omega_3}^q \quad - 24{\omega_3}^q P3(6{+}r) \qquad (65) \\ &\qquad \text{using } b^{-2}/(b^4{-}\omega_3) = -\tfrac{1}{24} \end{aligned}$$

In (65) the right hand terms are periodic in $r = 0, 1, 2, 3$ and $q = 0, 1, 2$. It uses the initial $P3(6)$ through $P3(9)$ which are calculated from the recurrences or by explicitly forming those hulls. The result is the 12 terms of $p(m)$.

$p(m)$ could be numbered starting anywhere mod 12. The choice here is to match the b power in each $P1$ etc. So the expression in (65) is reckoned as $p(k{+}2)$ to match its b^{k+2}.

$$p(k{+}2) \;=\; -b^{r+8}{\omega_3}^q \;-\; 24{\omega_3}^q\, P3(6{+}r) \qquad \square$$

From the coefficients of b^k in the point formulas, limits for extents of the curve scaled to a unit length are



A212952

Each hull vertex is a single-visited point. A double or triple-visited has 4 or 6 segments around it so is not a convex vertex. Point numbers n along the curve for each hull vertex follow from the sub-parts similar to the locations.

$$\begin{aligned}
&\mathit{PN1}(k) = \mathit{PN2}(k{-}1) && \mathit{PN5}(k) = \mathit{PN6}(k{-}1)\\
&\mathit{PN2}(k) = \mathit{PN3}(k{-}1) && \mathit{PN6}(k) = 3^{k-1} + \mathit{PN3}(k{-}1)\\
&\mathit{PN3}(k) = \mathit{PN4}(k{-}1) && \mathit{PN7}(k) = 3^{k-1} + \mathit{PN4}(k{-}1)\\
&\mathit{PN4}(k) = \mathit{PN5}(k{-}1) &&
\end{aligned}$$

P3B and P4B are the middle sub-part (ternary digit 1) so add 3^{k-1} in PN6 and PN7. Initial values at $k{=}6$ determine the low digits and then the 4-cycle P3–P4–P5–P6 is a high repeating pattern 1000. It's convenient to take that pattern as high 1 then repeat 0001 zero or more times, so as to simplify the low digit forms.

$$\begin{aligned}
\mathit{PN1}(k) &= \tfrac{1}{720}3^k + \tfrac{1}{80}[-9,\, 53,\, -1,\, -3]\\
&= \text{ternary } 1\ 0001\ 0001 \ldots\ \text{empty}, 0, 00\,\text{or}\,001 \quad \text{for } k{-}5 \text{ digits}\\
&= 1, 3, 9, 28, 82, 246, 738, 2215, 6643, \ldots \qquad k \ge 6
\end{aligned}$$

$$\begin{aligned}
&\mathit{PN2}(k) = \mathit{PN1}(k{+}1) && \mathit{PN4}(k) = \mathit{PN1}(k{+}3) && \mathit{PN6}(k) = \mathit{PN1}(k{+}5)\\
&\mathit{PN3}(k) = \mathit{PN1}(k{+}2) && \mathit{PN5}(k) = \mathit{PN1}(k{+}4) &&
\end{aligned}$$

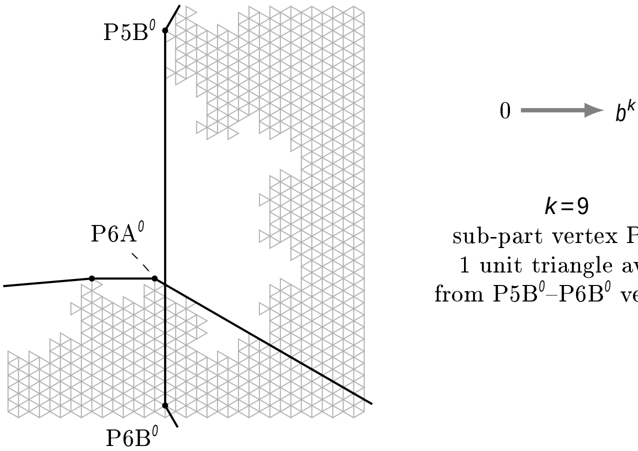
$$\begin{aligned}
\mathit{PN7}(k) &= \tfrac{83}{240}3^k + \tfrac{1}{80}[-1,\, -3,\, -9,\, 53]\\
&= \text{ternary } 1001\ 0001\ 0001 \ldots\ \text{empty}, 0, 00\,\text{or}\,001 \quad \text{for } k \text{ digits}\\
&= 252, 757, 2269, 6807, 20421, 61264, \ldots \qquad k \ge 6
\end{aligned}$$

In *PN7* the 3^{k-1} high ternary 1 digit is only 3 places above the rest of $\mathit{PN4}(k{-}1)$ so an initial 100 before the 1000 pattern.

In figure 17, the A sub-part vertex P6A′ is close to the B sub-part vertical P5B′ to P6B′. The vertex is on the line for $k \equiv 0, 2, 3 \bmod 4$ but is 1 unit triangle to the left when $k \equiv 1 \bmod 4$.

$$P6A'(k) = P6'(k{-}1) \qquad P6B'(k) = b^{k-1} + \omega_3 P6'(k{-}1)$$

$$\begin{aligned} \operatorname{Re} \frac{P6A'(k) - P6B'(k)}{{\omega_{12}}^k} &= \operatorname{Re} \tfrac{1}{24} \frac{p(k{+}4) - \omega_3 p(k{+}4)}{{\omega_{12}}^k} \\ &= \begin{cases} 0 & \text{if } k \equiv 0,2,3 \bmod 4 \\ -\frac{1}{2}\sqrt{3} & \text{if } k \equiv 1 \quad\ \bmod 4 \end{cases} \end{aligned}$$



$k{=}9$
sub-part vertex P6A^0
1 unit triangle away
from P5B^0–P6B^0 vertical

The area of the hull can be calculated taking consecutive points $P1, P2$ etc as triangles. The area of such a triangle is $\frac{1}{2}\operatorname{Im}\ z_1.\overline{z_2}$ in the usual way for z_1 to z_2 anti-clockwise around. Multiplying the vertex terms gives

$$\begin{aligned} \mathit{HA}_k &= \tfrac{\sqrt{3}}{4}. \begin{cases} 0, 2, & \text{if } k = 0, 1 \\ \frac{29}{24} 3^k - \frac{1}{12}[15, 23, 11, 25].3^{\lfloor k/2 \rfloor} - \frac{1}{8}[5, 3, 1, 3] & \text{if } k \geq 2 \end{cases} \\ &= \tfrac{\sqrt{3}}{4} \quad 0, 2, 8, 26, 86, 276, 856, 2586, \ldots \end{aligned}$$

The area of hulls $k = 0, 1$ are calculated explicitly. For $k \geq 2$ the duplications and extra vertices on the hull boundary give empty or split triangles but the general formula still applies.

Factor $\sqrt{3}/4$ is the area of a unit side equilateral triangle. It's convenient to write that for the $\sqrt{3}$.

Scaled by 3^k for start to end a unit length the hull area limit is

$$\frac{\mathit{HA}_k}{3^k} \to \tfrac{\sqrt{3}}{4}.\frac{29}{24} = 0.523223\ldots$$

Hull area $\frac{29}{24} = 1.208333\ldots$ equilateral triangles can be compared to the similar limit for the enclosed area $A_k/3^k \to \frac{2}{3} = 0.666\ldots$ equilateral triangles. The empty area enclosed by the hull is $\frac{29}{24} - \frac{2}{3} = \frac{13}{24} = 0.541666\ldots$.

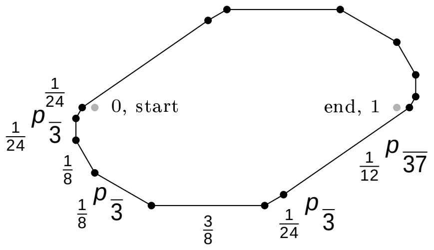
The hull boundary length, calculated from sides $|P1(k){-}P2(k)| + \cdots$ is

$$HB_k = \begin{cases} 2, 4 & \text{if } k = 0,1 \\ \left(\frac{13}{12}+\frac{5}{12}\sqrt{3}\right)\sqrt{3}^k & \text{if } k \ge 2 \\ +\frac{1}{6}\sqrt{37.3^k + [-30,162,30,-162].3^{\lfloor k/2 \rfloor} + [9,63]} & \\ +\ \left[\frac{9}{4}-\frac{7}{4}\sqrt{3},\ \frac{3}{4}-\frac{7}{4}\sqrt{3},\ \frac{3}{4}-\frac{5}{4}\sqrt{3},\ \frac{9}{4}-\frac{5}{4}\sqrt{3}\right] & \end{cases}$$

$$= 2,\ 4,\ 4{+}2\sqrt{3},\ 10{+}2\sqrt{3},\ 12{+}2\sqrt{3}{+}2\sqrt{19},\ 12{+}8\sqrt{3}{+}2\sqrt{73},\dots$$

The middle root term arises from sides $P7$–$P1'$ and $P7'$–$P1$ which are not at 30° angles. Scaled by $\sqrt{3}^k$ for start to end a unit length the limit is

$$\frac{HB_k}{\sqrt{3}^k} \to \frac{13}{12} + \frac{5}{12}\sqrt{3} + \frac{1}{6}\sqrt{37} = 2.818814\dots$$



hull boundary
side length
limits

Theorem 33. $P3$
$P3'$ k

$$HD_k = \sqrt[2]{\frac{21}{16}3^k - \frac{1}{8}[3,9,9,15].3^{\frac{k}{2}} + \frac{1}{16}[1,3,9,19]} \tag{66}$$

$$= \sqrt[2]{1,3,9,31,103,309,927,2821,\dots}$$

The points furthest apart must be vertices of the convex hull. For $k < 9$ the maximum distance points can be verified explicitly and are per the formula.

For $k \ge 9$, points $P1$ through $P7'$ of the convex hull are at various factors of b^k and offsets $p(m)$ from those powers. The offsets are at most

$$pmax = \max\left(\frac{1}{24}|p(m)|\right) = \frac{1}{8}\sqrt{19}$$

Comparing factors of b^k on the hull vertices, P3–P3′ are the furthest apart. Their distance is at least

$$|P3(k) - P3'(k)| \ge b^k + 2\frac{1}{24}b^{k+2} - 2pmax = \frac{1}{4}\sqrt{21}.\sqrt{3}^k - 2pmax$$

The second furthest by b^k factors is P2–P2′ and their distance, or the distance of any pair with smaller b^k factor, is at most

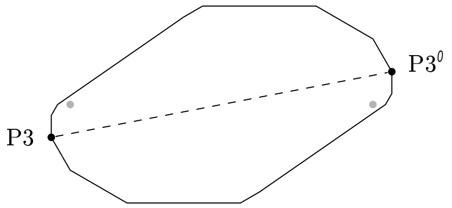
$$|P2(k) - P2'(k)| \le b^k + 2\frac{1}{24}b^{k+1} + 2pmax = \frac{1}{4}\sqrt{\frac{61}{3}}.\sqrt{3}^k + 2pmax$$

For $k \ge 9$ the difference between the two bounds is positive, as seen by decreasing and increasing terms to convenient squares,

$$\frac{1}{4}\sqrt{21} - \frac{1}{4}\sqrt{\frac{61}{3}}\ \sqrt{3}^9 - 4pmax \qquad k \ge 9$$

$$> \tfrac{1}{4}\sqrt{\tfrac{458^2}{10000}} - \tfrac{1}{4}\sqrt{\tfrac{451^2}{10000}}\ 140 - 4\tfrac{1}{8}\sqrt{\tfrac{436^2}{10000}} \quad = \tfrac{27}{100} > 0 \qquad \square$$

Scaled by $\sqrt{3}^k$ for start to end a unit length the distance is square root of the coefficient of the 3^k term in (66).



curve points
furthest apart

$$\frac{HD_k}{\sqrt{3}^k} \rightarrow \frac{\sqrt{21}}{4} = 1.145643\ldots$$

HD is between any two points of the curve. It's also possible to consider only points on lines parallel to curve start to end.

Theorem 34.

$$P1S \qquad P1S'$$

$$P1S_k = \begin{cases} 0 & k = 0,1 \\ P1(k) + \omega_{12}^{k+1} & k \equiv 1 \bmod 4 \qquad k \geq 5 \\ P1(k) & \end{cases}$$

$$P1S'_k = b^k - P1S_k$$

$$\begin{aligned} HSD_k &= |P1S_k - P1S'_k| \\ &= \begin{cases} 1, \sqrt{3} & k = 0,1 \\ \frac{13}{12}\sqrt{3}^k - [\frac{3}{4}, \frac{3}{4}\sqrt{3}, \frac{1}{4}, \frac{1}{4}\sqrt{3}] & k \geq 1 \end{cases} \\ &= 1, \sqrt{3},\ 3, 3\sqrt{3},\ 9, 9\sqrt{3},\ 29, 29\sqrt{3},\ 87, 87\sqrt{3}, \ldots \end{aligned} \tag{67}$$

$k = 2m$ $\quad$ 10 0202... $\quad$ $m-1$ $\quad$ $k = 2m+1$

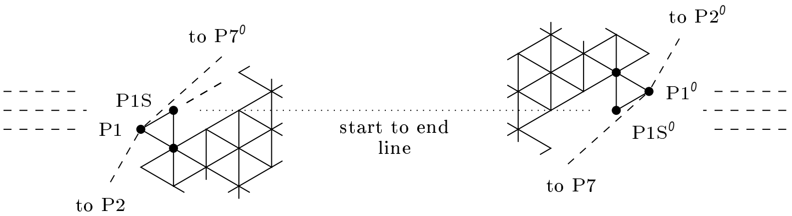
$\sqrt{3}$

Greatest distances can be verified explicitly for $k \leq 6$. For $k \geq 7$, hull vertex P1 is on the start to end line when $k \not\equiv 1 \bmod 4$ since its formula has

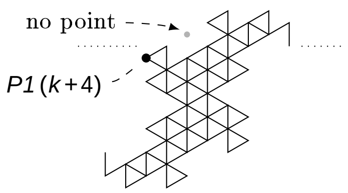
$$\operatorname{Im} P1(k)/\omega_{12}^k = [0, -\tfrac{1}{2}, 0, 0]$$

Hull side P1–P2 is at 60° to the line start to end and side P1–P7′ is at less than 60°, so any parallel points away from P1 are shorter than P1 to P1′.

For $k \equiv 1 \bmod 4$, in the $k-1$ hulls of figure 17, P1 = P2A has adjacent sides 60° and 30° so that anywhere other the overlap arising from $\operatorname{Im} P1(k) = -\frac{1}{2}$ is shorter.



P1 is at 30° down from P1S. Distance P1S to P1S′ could be equalled by P1 to a point below left 30° from P1S′. Or likewise from P1′ to a point above right 30° of P1S. But these points are not in the curve. They are not in k=9 and thereafter the P1–P1S segment expands 4 times as follows for new $\mathit{P1}\,(k{+}4)$ also without point above right.



k 1 mod 4

☐

For the curve scaled to a unit length, the limit is the distance P1 to P1′ which is the coefficient of $\sqrt{3}^k$ in (67),

$$\frac{\mathit{HSD}\,(k)}{b^k} \to \frac{13}{12}$$

A maximum distance between two points on a line perpendicular to start to end is the corresponding points in the middle third of the curve, so $\mathit{P1S}\,(k{-}1)$ and $\mathit{P1S}'(k{-}1)$ in the $k{-}1$ middle part of figure 17. These points are not on the whole curve hull boundary. Their width limit is simply $/\sqrt{3}$,

$$\frac{\mathit{HSD}\,(k{-}1)}{b^k} \to \frac{13\sqrt{3}}{36} = 0.625462\ldots$$

Theorem 35.
$\frac{1}{2}b^k$

$$\mathit{Lnear}_k = \begin{cases} 0 \text{ and } 1 & k{=}0 \\ 1 \text{ and } \frac{1}{2}{+}\frac{1}{2}\sqrt{3}i & k{=}1 \\ \frac{1}{2}{+}\frac{1}{2}\sqrt{3}i \text{ and } 1{+}\sqrt{3}i & k{=}2 \\ \dfrac{19+\sqrt{3}i}{48}\,b^k + \frac{1}{24}\mathit{pt}\,(k{+}1) & k \ge 3 \\ \qquad k{=}5 & \quad -\frac{11}{2}{+}\frac{3}{2}\sqrt{3}i \end{cases}$$

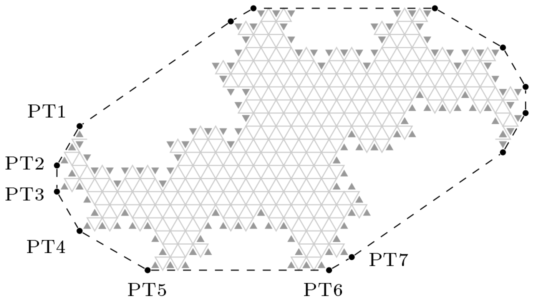
$$\begin{aligned} \mathit{pt}\,(m) = [\,15, &\quad 6{-}9\omega_3, \quad -3{-}27\omega_3, \quad -3{+}18\omega_3, \\ 15\omega_3, &\quad (6{-}9\omega_3)\omega_3, \quad (-3{-}27\omega_3)\omega_3, \quad (-3{+}18\omega_3)\omega_3, \end{aligned} \tag{68}$$

$$15\omega_3^{\,2},\ \ (6-9\omega_3)\omega_3^{\,2},\ \ (-3-27\omega_3)\omega_3^{\,2},\ \ (-3+18\omega_3)\omega_3^{\,2}\ ]$$

$$\mathit{Rnear}_k = b^k - \mathit{Lnear}_k$$

For $k \le 5$ the points nearest the middle can be calculated explicitly.

For $k \ge 6$, boundary points correspond to corners of triangles on the boundary of surrounding curves. Form the convex hull around segments plus boundary triangles. This can be calculated the same as the segments hull in theorem 32, since the curve with boundary triangles is an unfold of sub-curves $k{-}1$ and their boundary triangles. For $k \ge 6$ there are 14 vertices (like the segments hull).



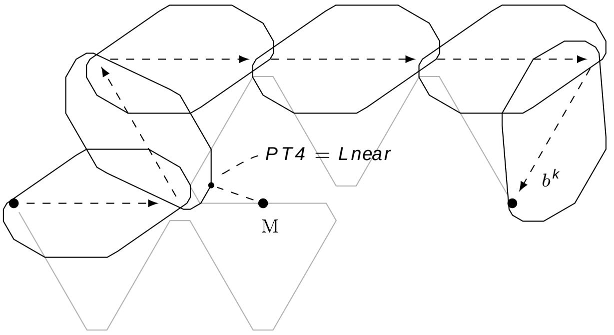
k = 6
convex hull
segments and
boundary triangles

The boundary triangles push the segments hull vertices out by 1 unit triangle on each straight side. The triangles hull vertices are at corners of a triangle, since a curve point would have triangles each side of it and so not be a hull vertex.

Working through the hull recurrences the result is the same location forms as segments P1 etc (64), but different offset terms. Each p in P1 etc becomes pt at (68) in PT1 etc.

$$\mathit{PT1}(k) = -\tfrac{1}{24}\ \ b^k + \mathit{pt}(k) \qquad \text{etc}$$

Consider then curve k comprising $k{-}2$ sub-curves and surrounding $k{-}2$ sub-curves. The triangle hulls around those surrounding sub-curves are



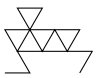


Figure 18:
k 2 hulls
surrounding
left side

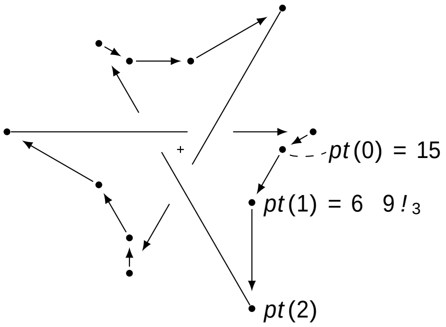
The boundary triangles push into the left boundary L so that minimum extents for the left boundary points are given by maximum extents of the surrounding hulls.

The claimed *Lnear* is the marked *PT4* in figure 18, being *PT4* in that surrounding $k{-}2$ hull. Its sub-curve starts at $\frac{1}{3}b^k$. Its subcurve endpoint b^{k-2} is directed $-60°$ relative to the b^k end. So $+120°$ direction in figure 18 is total turn $180°$ so negate,

$$\mathit{Lnear}_k = \tfrac{1}{3}b^k - \mathit{PT4}(k{-}2)$$

Working through the hull formulas it can be verified that this is nearer than the other hulls, and that the slopes of the sides adjacent to *PT4* are more than $90°$ to a line M–*PT4* so that nothing else in the surrounding hull is nearer. ☐

The offsets in *pt* can be illustrated



The difference between p and *pt* is effectively which sides are pushed out by the boundary triangles in the way noted above.

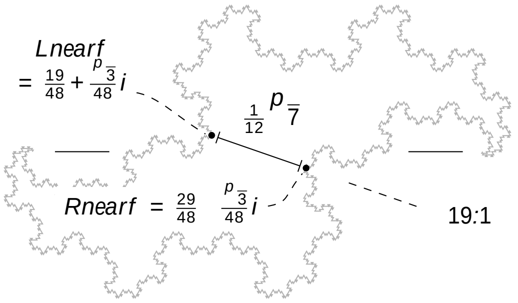
$$\begin{aligned}\tfrac{1}{24}(\mathit{pt}(k) - p(k)) &= [1,\ -\omega_3, -\omega_3, \omega_3, \omega_3,\ -\omega_3^2, -\omega_3^2, \omega_3^2, \omega_3^2,\ -1, -1, 1]\\ &= \omega_3^{\lfloor (k+3)/4 \rfloor}.(-1)^{\lfloor (k+1)/2 \rfloor}\end{aligned}$$

For endpoints scaled to a unit length, the limits for *Lnear* and *Rnear* are their b^k coefficients.

$$\begin{aligned}\frac{\mathit{Lnear}_k}{b^k} &\to \mathit{Lnearf} = \frac{19+\sqrt{3}i}{48} = \frac{10+\omega_3}{24} = 0.3958333\ldots + 0.036084\ldots i\\ \frac{\mathit{Rnear}_k}{b^k} &\to \mathit{Rnearf} = \frac{29-\sqrt{3}i}{48} = \frac{14-\omega_3}{24} = 0.6041666\ldots - 0.036084\ldots i\end{aligned}$$

A line between *Lnearf* and *Rnearf* is the narrowest part through the middle. The length of that line and the angle down from the curve start to end are

$$\begin{aligned}|\mathit{Rnearf} - \mathit{Lnear}| &= \tfrac{1}{12}\sqrt{7} = 0.220479\ldots\\ \arg(\mathit{Rnearf} - \mathit{Lnear}) &= -\arctan\tfrac{1}{5}\sqrt{3} = -19.106605°\ldots\end{aligned}$$



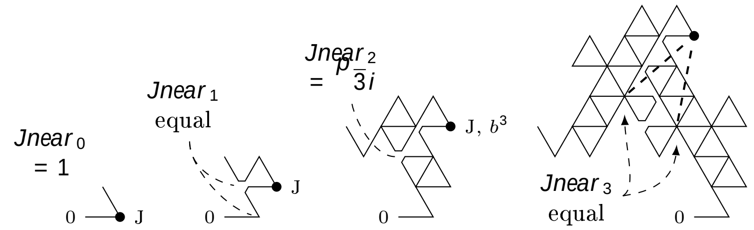
Lnear; Rnear limits, width and angle

Theorem 36. 60°

$$
\mathit{Jnear}_k = \begin{cases} 1 & & & k{=}0 \\ 1 & \omega_6 & & k{=}1 \\ \sqrt{3}i & & & k{=}2 \\ \mathit{JnearPT2}_k & & \mathit{JnearPT2}_k - \bar{b} & k{=}3 \\ \mathit{JnearPT2}_k & & \mathit{JnearPT2}_k - \omega_6 & k{=}6 \\ \mathit{JnearPT2}_k \end{cases}
$$

$$
\begin{aligned} \mathit{JnearPT2}_k &= \omega_6 \quad b^{k-1} + \omega_3 \mathit{PT2}(k{-}1) \\ &= \ \tfrac{13}{24} + \tfrac{1}{6}\sqrt{3}i \quad b^k \ + \tfrac{1}{24}\mathit{pt}(k) \end{aligned} \tag{69}
$$

The nearest points for $k \leq 6$ can be calculated explicitly.



The $k{-}1$ triangle hulls of the absent left k side are

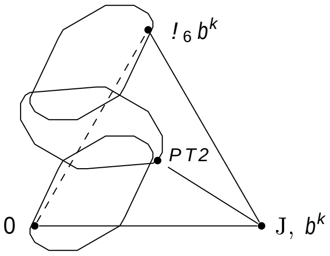
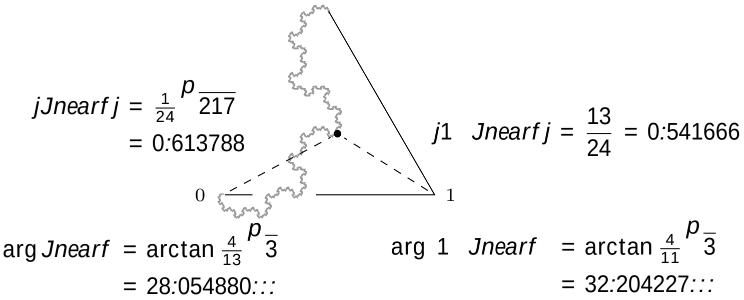


Figure 19: absent k 1 hulls

Working through the formulas shows the nearest to J is *PT2* of the middle hull. Its adjacent sides are 30° before and 60° after which are past 90° perpendicular to the line from J, so other points are further away. ☐

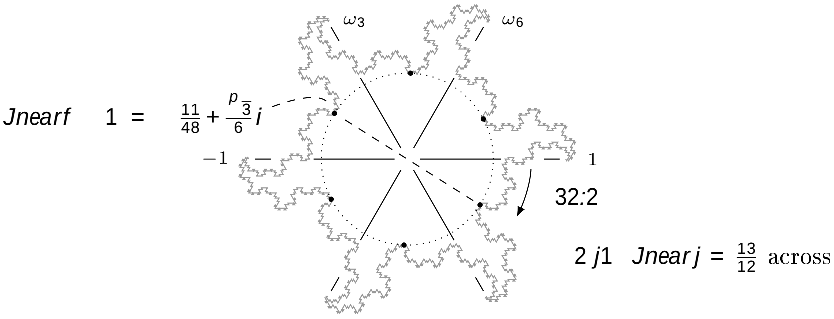
The limit for join of curves scaled to unit lengths is the b^k coefficient in (69).

$$\frac{\mathit{Jnear}_k}{b^k} \to \mathit{Jnearf} = \tfrac{13}{24} + \tfrac{1}{6}\sqrt{3}\,i \;= \tfrac{17}{24} + \tfrac{1}{3}\omega_3$$
$$= 0.541666\ldots + 0.288675\ldots i \qquad \text{imag A020769}$$



Jnearf is located $+\frac{1}{24}$ right of the middle $\frac{1}{2}+\frac{1}{6}\sqrt{3}i = b/3$. The middle is a sub-curve endpoint with two sub-curves. Those absent sub-curves (the first two hulls in figure 19) spiral around that middle, as all curve ends do, giving boundary points closer to J than the middle.

Jnearf $-\,1$ is the narrowest part through the middle of 6 arm plane filling per section 7 (and by symmetry the same at successive 60°).



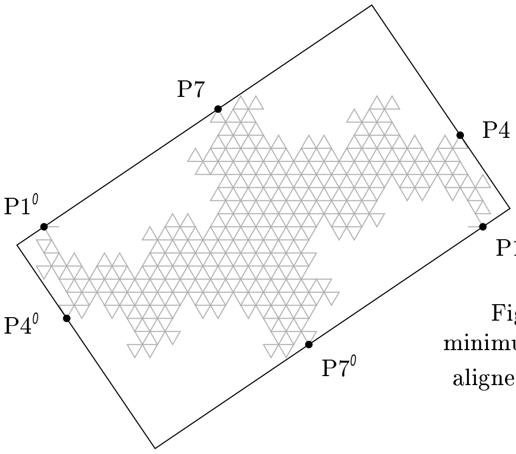


Figure 20: $k=6$
minimum area rectangle
aligned to side P7–P1′

Theorem 37. k

$$MR_k = \frac{\sqrt{3}}{4}\begin{cases} 0, 3, 9, 33 & k = 0 \text{ to } 3 \\ \dfrac{MrW_k . MrH_k}{MrDen_k} & k \geq 4 \end{cases} \tag{70}$$

$$= \frac{\sqrt{3}}{4}\; 0, 3, 9, 33, \tfrac{2187}{19}, \tfrac{25392}{73}, \tfrac{205407}{193}, \ldots$$

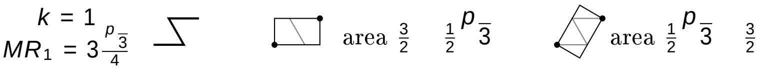
$$\begin{aligned} MrW_k &= \tfrac{13}{12}3^k + \tfrac{1}{12}[-9, 18, -1, -30].3^{\lfloor k/2 \rfloor} + \tfrac{1}{4}[0, -3, -2, 1] \\ &= 81, 276, 787, 2302, 7047, 21444, \ldots \qquad k \geq 4 \\ MrH_k &= \tfrac{13}{36}3^k + \tfrac{1}{12}[-3, 6, -1, -16].3^{\lfloor k/2 \rfloor} + \tfrac{1}{4}[0, -1, 0, 1] \\ &= 27, 92, 261, 754, 2349, 7148, \ldots \qquad k \geq 4 \\ MrDen_k &= |P7(k) - P1'(k)|^2 \\ &= \tfrac{37}{144}3^k + \tfrac{1}{24}[-5, 27, 5, -27].3^{\lfloor k/2 \rfloor} + \tfrac{1}{16}[1, 7] \\ &= 19, 73, 193, 532, 1669, 5149, \ldots \qquad k \geq 4 \end{aligned}$$

$k \geq 4$ $'$ $k = 1$ to 3
$+30°$ k=0

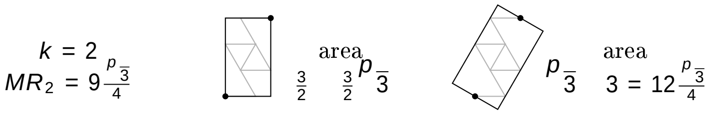
A minimum area rectangle has at least one side aligned to a side of the convex hull, so it suffices to consider rectangles on the hull sides.

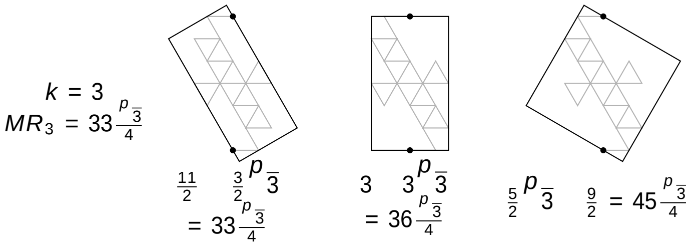
For k=0 the curve is a unit line segment with area $MR_0 = 0$.

For k=1 the two possible rectangle alignments both have area $MR_1 = 3\frac{\sqrt{3}}{4}$.



For k=2 and k=3 the possible alignments and areas are as follows. In each case the first is the minimum and is per the general formula.

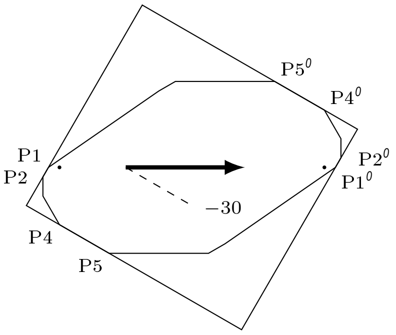




For $k \geq 4$ the hull vertices $P1(k)$ through $P7(k)$ from the convex hull (64) can be used. The formulas are used for k=4 and k=5 since as in the diagrams above those formulas are all hull vertices (with some repetitions).

There are 7 sides (and 180° reversals). The first 6 are 30° turns which means 90° after the first 3, so total 4 distinct alignments.

A rectangle aligned $-30°$ to the b^k endpoint, which is the P1–P2 and P4–P5 sides for $k{\ge}6$, is



It's convenient to divide by b^{k+2} to rotate with a factor $3^{k+2} = |b^{k+2}|^2$ to scale back to unit length segments. P2–P2′ and P5–P5′ are suitable extents for $k \ge 1$,

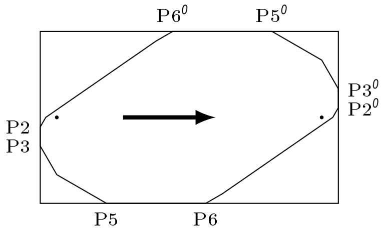
$$\mathit{MR12}\,(k) = 3^{k+2}.\,\mathrm{Re}\,\frac{P5'(k) - P5(k)}{b^{k+2}}.\,\mathrm{Im}\,\frac{P2(k) - P2'(k)}{b^{k+2}} \qquad k \ge 1$$

$P2$ is used here rather than $P1$ since $P1(1)$ is not on the hull boundary, although actually its extents are the same as $P2$ there. Then with a $-30°$ hull explicitly calculated around the $k{=}0$ line segment for completeness,

$$\mathit{MR12}\,(k) = \tfrac{\sqrt{3}}{4} \; \begin{cases} 1 & \text{if } k{=}0 \\ \tfrac{91}{48}3^k - \tfrac{1}{24}[51,69,17,51].3^{\;\frac{k}{2}} \; + \tfrac{1}{16}[9,3,1,3] & \text{if } k \ge 1 \end{cases}$$

$$= \tfrac{\sqrt{3}}{4} \;\; 1, 3, 15, 45, 135, 435, \ldots \qquad k \ge 0$$

A rectangle aligned to the b^k endpoint, which is sides P2–P3 and P5–P6 sides for $k{\ge}5$, is

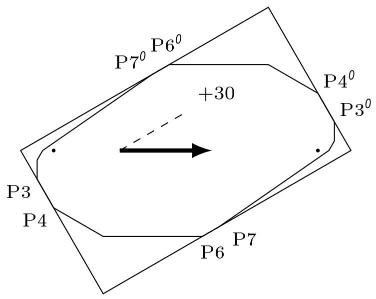


$$\mathit{MR23}\,(k) = 3^k.\,\mathrm{Re}\,\frac{P3(k) - P3'(k)}{b^k}.\,\mathrm{Im}\,\frac{P5(k) - P5'(k)}{b^k}$$

$$= \tfrac{\sqrt{3}}{4} \;\; \tfrac{27}{16}3^k - \tfrac{1}{8} \;\; 15, 9, 9, 27 \;\; .3^{\;\frac{k}{2}} \;\; + \tfrac{1}{16} \;\; 3, 1, 3, 9$$

$$= \sqrt{3} \;\; 0, 1, 3, 9, 30, 100, \ldots \qquad k \ge 0$$

A rectangle aligned +30° to the b^k endpoint, which is the P3–P4 and P6–P7 sides for $k \geq 4$, is



$$\mathit{MR34}\,(k) = 3^{k+1}.\,\mathrm{Re}\,\frac{P3(k) - P3'(k)}{b^{k+1}}.\,\mathrm{Im}\,\frac{P6(k) - P6'(k)}{b^{k+1}}$$

$$= \tfrac{\sqrt{3}}{4}\quad \tfrac{25}{16}.3^k - \tfrac{1}{8}\;\; 5, 15, 15, 25\;\; .3^{\;\frac{k}{2}}\;\; + \tfrac{1}{16}\;\; 1, 3, 9, 3$$

$$= \tfrac{\sqrt{3}}{4}\quad 1, 3, 9, 33, 121, 363, \ldots \qquad\qquad k \geq 0$$

MR34 is the alignment of the minimum area rectangles around $k = 1\,\mathrm{to}\,3$ as above.

The final alignment is to the P7–P1′ side. That side turns away from P6–P7 and since P3–P4 is at 90° to P6–P7 the P4 vertex is the rectangle width, as shown in the sample figure 20.

$$\mathit{MR71}\,(k) = \;\; P7(k) - P1'(k)\;\;^2.\,\mathrm{Re}\,\frac{P4(k) - P4'(k)}{P7(k) - P1'(k)}\,.\,\mathrm{Im}\,\frac{P7(k) - P7'(k)}{P7(k) - P1'(k)}$$

$$= \frac{\mathrm{Re}\,(P4{-}P4')(\overline{P7{-}P1'})\,.\,\mathrm{Im}\,(P7{-}P7')\overline{(P7{-}P1')}}{|P7 - P1'|^2}$$

The numerator "width" and "height" at (70) are the respective real and imaginary parts but with factor $\sqrt{3}/4$ taken out.

$$\mathit{MrW}_k = \tfrac{1}{2}\,\mathrm{Re}\,(P4{-}P4')\,\overline{(P7{-}P1')}$$
$$\mathit{MrH}_k = \tfrac{\sqrt{3}}{2}\,\mathrm{Im}\,(P7{-}P7')\,\overline{(P7{-}P1')}$$

To compare to *MR12* etc divide down to

$$\mathit{MR71}\,(k) = \tfrac{\sqrt{3}}{4}\quad \tfrac{169}{111}3^k - a(k)\,3^{\lfloor k/2 \rfloor} + \frac{b(k)\,3^k + c(k)\,3^{\lfloor k/2 \rfloor} + d(k)}{\mathit{MrDen}_k}$$
$$a(k) = \;\tfrac{1196}{1369}, \tfrac{3354}{1369}, \tfrac{6994}{4107}, \tfrac{3380}{1369}$$
$$b(k) = \;-\tfrac{491}{5476}, -\tfrac{855}{21904}, \tfrac{471}{5476}, -\tfrac{23605}{197136}$$
$$c(k) = \;\tfrac{299}{5476}, \tfrac{3525}{10952}, \tfrac{811}{5476}, \tfrac{4003}{32856}$$
$$d(k) = \;0, \tfrac{3}{16}, 0, \tfrac{1}{16}$$

Factor $\frac{169}{111}$ on 3^k in *MR71* is smaller than the corresponding $\frac{91}{48}$, $\frac{27}{16}$ and $\frac{25}{16}$ of the other alignments. For $k \geq 4$ the difference exceeds the half-power and constant terms and so is the minimum area rectangle. □

11 Moment of Inertia

The mass moment of inertia $I = \sum mr^2$ of a rigid body rotating around a given axis is the ratio of torque to angular acceleration, similar to the way ordinary mass is the ratio of force to linear acceleration.

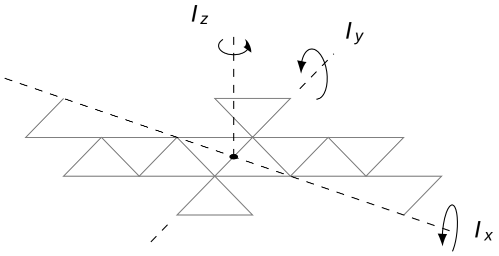


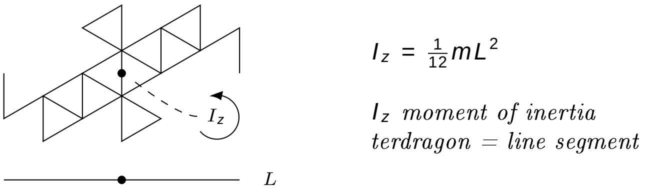
Figure 21: moment of inertia

Rotating about the z axis keeps the curve within the plane. This case is the simplest.

Theorem 38.

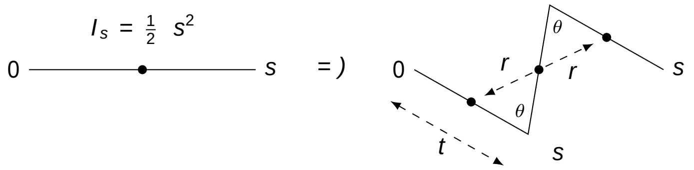
θ

I_z



For k=0 the curve is a straight line so the statement is true.

Suppose the statement is true of level k. Let each of its segments have mass μ and length s. The moment of inertia of such a segment about its centre is $I = \frac{1}{12}\mu s^2$. In the next expansion the segment unfolds by angle θ as follows



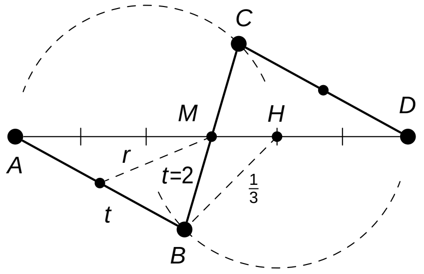
There are now 3 segments each length t and mass $\mu/3$. The centre of mass is unchanged. The moment of inertia I' of the expanded shape about this centre is also unchanged since

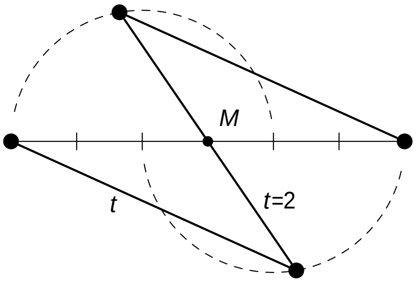
$\beta = 1/(2 + e^{i(\pi-\theta)})$	reduction
$t = s\,\lvert\beta\rvert$	new segment length
$r = s\ \frac{1}{2} - \frac{1}{2}\beta$	to midpoints

$$
\begin{aligned}
I' &= 3\frac{1}{12}\frac{\mu}{3}\,t^2 \;+\; 2\,\frac{\mu}{3}\,r^2 && \text{parallel axis theorem} && (71)\\
&= \frac{1}{12}\,\mu s^2 \;\; |\beta|^2 + 2\,|1{-}\beta|^2 \\
&= \frac{1}{12}\,\mu s^2 \; \frac{1+2\;\; 1+\cos(\pi{-}\theta)\;\;^2 + \sin^2(\pi{-}\theta)}{2+\cos(\pi{-}\theta)\;\;^2+\sin^2(\pi{-}\theta)} && && (72)\\
&= \frac{1}{12}\,\mu s^2 \quad = \; I && && \square
\end{aligned}
$$

The usual terdragon is $\theta = 60°$. It has $t = s/\sqrt{3}$ and the triangle formed by r is equilateral so $r = t/2$. Applying this to (71) easily gives $I' = I$. For other angles r and t vary inversely and the sin and cos terms of (72) cancel out so $I' = I$ always.

The following diagram shows the geometry of the expansion. $\overline{AD}$ is a fixed length. $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$ are the three new line segments each length t. B is distance $t/2$ from the middle M.



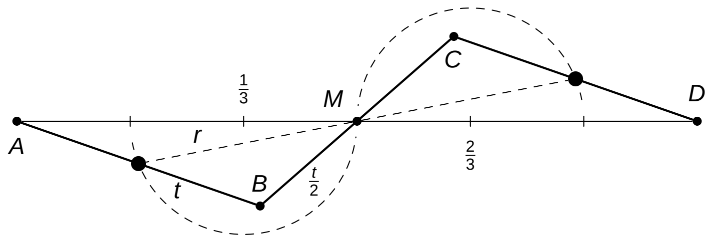


H is at $\frac{2}{3}$ along $\overline{AD}$. The distance $\overline{HB}$ is

$$\overline{HB} = \;\tfrac{2}{3}s - bs\; = \frac{1}{3}s \;\sqrt{\frac{(1+2\cos)^2+(2\sin)^2}{(2+\cos)^2+\sin^2}} = \frac{1}{3}s$$

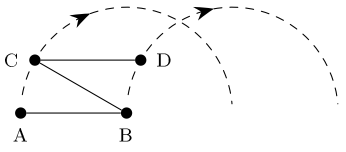
so B is on a circle of radius $\frac{1}{3}$ centred at H. Likewise by symmetry C on the corresponding circle above.

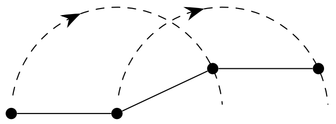
The midpoint of $\overline{AB}$, which is where r measures to, also follows a circle as in the following diagram. This is simply because the $\overline{AB}$ midpoint follows the circle of B but shrunk by $\frac{1}{2}$ in both x and y directions. So where B arcs from $\frac{1}{3}$ to 1 the $\overline{AB}$ midpoint arcs from $\frac{1}{6}$ to $\frac{1}{2}$.



The first circle is centred at $\frac{1}{3}$ with radius $\frac{1}{6}$. Similarly the corresponding upper arc. The two meet at M since both $\overline{AB}$ and $\overline{CD}$ midpoints are in the middle when fully overlapping $\overline{AB} = \overline{CD} = \overline{AD}$ for no unfold θ=0.

The points also make circles when the line segments $\overline{AB}$ etc are fixed lengths. This is obvious for C since it pivots from B. D is a fixed offset to the right so is a shift of the C circle.





Theorem 39.

x

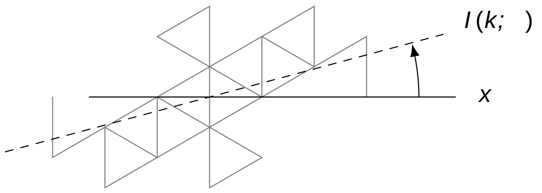
$$\begin{pmatrix} I_x & -I_{xy} & 0 \\ -I_{xy} & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \qquad \begin{array}{ll} I_x = \sum y^2 & I_{xy} = \sum xy \\ I_y = \sum x^2 & I_z = \sum x^2+y^2 \end{array}$$

$$\begin{aligned} I_x(k) &= \tfrac{1}{84}\ 2.9^k - [2,-3].(-3)^{\lfloor k/2 \rfloor} \\ &= 0,\ \tfrac{1}{4},\ 2,\ \tfrac{69}{4},\ 156,\ \tfrac{5625}{4},\ 12654,\ \tfrac{455517}{4},\ \ldots \\ I_y(k) &= \tfrac{1}{84}\ 5.9^k + [2,-3].(-3)^{\lfloor k/2 \rfloor} \\ &= \tfrac{1}{12},\ \tfrac{1}{2},\ \tfrac{19}{4},\ \tfrac{87}{2},\ \tfrac{1563}{4},\ \tfrac{7029}{2},\ \tfrac{126531}{4},\ \tfrac{569403}{2},\ \ldots \\ I_{xy}(k) &= \tfrac{\sqrt{3}}{168}\ 2.9^k - [2,4].(-3)^{\lfloor k/2 \rfloor} \qquad (73) \\ &= \sqrt{3}.\ \left\{ 0,\ \tfrac{1}{12},\ 1,\ \tfrac{35}{4},\ 78,\ \tfrac{2811}{4},\ 6327,\ \tfrac{227763}{4},\ \ldots \right\} \\ I_z(k) &= I_x(k) + I_y(k) \ = \tfrac{1}{12} 9^k \\ &= \tfrac{1}{12},\ \tfrac{3}{4},\ \tfrac{27}{4},\ \tfrac{243}{4},\ \tfrac{2187}{4},\ \tfrac{19683}{4},\ \ldots \end{aligned}$$

$k{\ge}1$ $\frac{1}{4}$A013708

I_x and I_y are the moments of inertia rotating about the x or y axes as in figure 21. They can be combined with I_{xy} in the usual way for inertia about an axis at angle α in the plane

$$I(k,\alpha) \ = \ I_x(k).\cos^2\alpha \ - \ 2I_{xy}(k).\cos\alpha\sin\alpha \ + \ I_y(k).\sin^2\alpha$$



For $k{=}0$ the curve is a single line segment and that line has inertia $I_x(0) = 0$, $I_{xy}(0) = 0$ and $I_y(0) = \frac{1}{12}$ which is per the formulas.

For $k \geq 1$ the inertia is calculated by rotations and the parallel axis theorem from the 3 copies of level $k{-}1$.

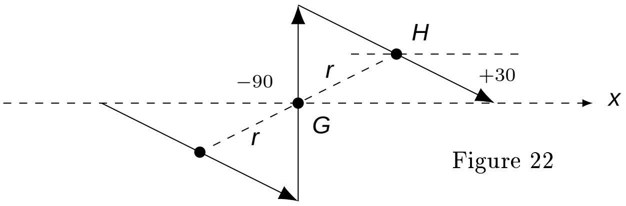


Figure 22

The first and last copies have the x axis at $+30°$ relative to those copies. The axes are turned by a matrix of rotation in the usual way

$$R = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{rotate axes by } +30°$$

Distance r is half the $k{-}1$ curve extent $r = \frac{1}{2}(\sqrt{3})^{k-1}$ and it is at $-30°$ to the axes for shifting the centre of mass of the first and last sub-curves. The middle sub-curve is axes at $-90°$. So total

$$\begin{aligned} I(k) &= 2\,R^{-1}.I(k{-}1).R && \text{first and last } +30° \\ &\quad + R^3.I(k{-}1).R^{-3} && \text{middle } -90° \\ &\quad + 2.3^{k-1}.\left(\tfrac{1}{2}\sqrt{3}^{k-1}\right)^2.R\begin{pmatrix}0&0\\0&1\end{pmatrix}R^{-1} && mr^2 \text{ first and last } -30° \end{aligned}$$

Multiplying through is mutual recurrences

$$I_x(k) = \tfrac{3}{2}I_x(k{-}1) - \sqrt{3}I_{xy}(k{-}1) + \tfrac{3}{2}I_y(k{-}1) + \tfrac{1}{8}9^{k-1} \tag{74}$$

$$I_y(k) = \tfrac{3}{2}I_x(k{-}1) + \sqrt{3}I_{xy}(k{-}1) + \tfrac{3}{2}I_y(k{-}1) + \tfrac{3}{8}9^{k-1} \tag{75}$$

$$I_{xy}(k) = \tfrac{1}{2}\sqrt{3}I_x(k{-}1) - \tfrac{1}{2}\sqrt{3}I_y(k{-}1) + \tfrac{1}{8}\sqrt{3}.9^{k-1}$$

I_{xy} has difference $I_x - I_y$ and subtracting (74)−(75) is that $I_x - I_y$ in terms of I_{xy} again so a recurrence for I_{xy} which can be expanded and summed down to either $I_{xy}(0)$ or $I_{xy}(1)$ according as k even or odd.

$$\begin{aligned} I_{xy}(k) &= -3\,I_{xy}(k{-}2) + \sqrt{3}.9^{k-2} \\ &= \sqrt{3}\,\frac{9^k - 9^{(k \bmod 2)}(-3)^{\lfloor k/2 \rfloor}}{81 - (-3)} + I_{xy}(k \bmod 2).(-3)^{\lfloor k/2 \rfloor} \\ &\qquad \text{where } k \bmod 2 \text{ means 0 or 1 as } k \text{ even or odd} \end{aligned}$$

With initial $I_{xy}(0) = 0$ and $I_{xy}(1) = \frac{1}{12}\sqrt{3}$ from the mutual recurrences (or explicit calculation) this gives (73).

I_z is equivalent to a straight line as from theorem 38. The line here is extent $(\sqrt{3})^k$ and mass 3^k so $I_z = \frac{1}{12}9^k$. $I_z = I_x{+}I_y$ for any plane figure. Substituting I_{xy} and $I_y = \frac{1}{12}9^k - I_x$ into (74) gives I_x, and from which I_y. □

Variations can be made with a different mass distribution on each line segment. For example a unit mass at the midpoint of each segment would make the initial $I_y(0)$ zero and change $I_{xy}(1)$ and the factor on $(-3)^{\lfloor k/2 \rfloor}$ in I_{xy}. Subtracting the individual line segments inertia $\frac{1}{12}3^k$ from I_z introduces a 3^k term into I_x and I_y.

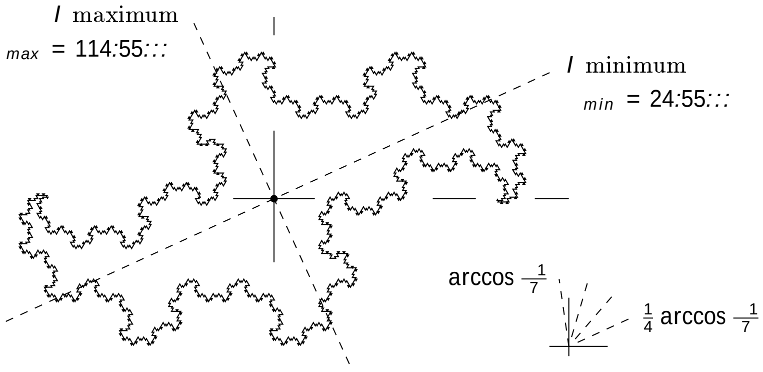
An inertia matrix is real and symmetric so can be diagonalized with a suitable matrix of rotation turning to the eigenvectors which are its principal axes. The physical significance of this is that rotation about those axes is perfectly balanced with no torque exerted on the mounting points.

In the usual way for a 2×2 matrix the eigenvectors are in direction d where

$$\begin{aligned}
d^2 &= \ I_x(k) - I_y(k) \ \ - \ 2I_{xy}(k)\, i \\
\alpha &= \tfrac{1}{2} \arctan \frac{-2I_{xy}(k)}{I_x(k) - I_y(k)} \ + (0 \text{ or } \tfrac{\pi}{2}) \\
&= \tfrac{1}{2} \arctan \ \tfrac{2}{\sqrt{3}} - \epsilon_k \ \ + (0 \text{ or } \tfrac{\pi}{2}) \\
\epsilon_k &= \begin{cases} \dfrac{14\sqrt{3}}{9.(-27)^{k/2} + 12} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}
\end{aligned}$$

$\epsilon_k \to 0$ as $k \to \infty$ so the limit for the principal axes is the $\frac{1}{2} \arctan \frac{2}{\sqrt{3}}$. A double-angle can square the hypotenuse $|2+\sqrt{3}i| = \sqrt{7}$ to an arccos if preferred.

$$\begin{aligned}
\alpha_{min} &\to \tfrac{1}{4} \arccos \tfrac{-1}{7} \ \ = 24.553302\ldots^\circ && \text{arccos second quadrant} \\
\alpha_{max} &= \alpha_{min} + \tfrac{\pi}{2} \\
&\quad \to \pi - \tfrac{1}{4} \arccos \tfrac{-1}{7} \ \ = 114.553302\ldots^\circ && \text{arccos third quadrant}
\end{aligned}$$



Roughly speaking, the minimum inertia is where the curve is closest to the axis and the maximum is where the curve is furthest from the axis, as measured by mr^2.

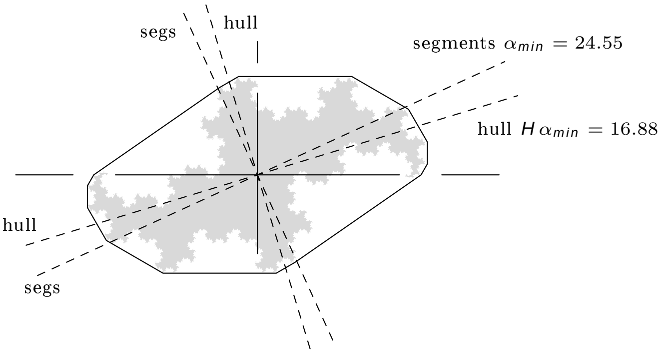
For the curve scaled to unit length, unit mass, and rotated $-\alpha_{min}$ so x axis minimum, the inertia limit is

$$\begin{pmatrix}
\frac{1}{24} - \frac{1}{168}\sqrt{21} & 0 & 0 \\
0 & \frac{1}{24} + \frac{1}{168}\sqrt{21} & 0 \\
0 & 0 & \frac{1}{12}
\end{pmatrix}$$

The inertia of the convex hull can be compared to that of the curve it surrounds. The inertia of the hull is calculated from its polygon. Its limit scaled to a unit length and with mass equal to its area is

$$\begin{aligned} \mathit{HI}_x &= \tfrac{7261}{884736}\sqrt{3} &&= 0.0142148\ldots \qquad \text{hull inertia} \\ \mathit{HI}_y &= \tfrac{58999}{2654208}\sqrt{3} &&= 0.0385008\ldots \\ \mathit{HI}_{xy} &= \tfrac{449}{55296} &&= 0.0081199\ldots \end{aligned}$$

$$H\alpha_{min} = \tfrac{1}{2}\arctan \ \tfrac{449}{1163}\sqrt{3} \quad = 16.885199^\circ\ldots$$



The segments axis α_{min} is close to hull vertex P4 but does not pass through it since P4 is at a slightly smaller slope,

$$\begin{aligned} &\frac{\mathit{P4}(k)}{b^k} \to \mathit{P4f} = -\tfrac{1}{8}\sqrt{3}\,i \\ &\arg\ \tfrac{1}{2} - \mathit{P4f} \quad = \arctan \tfrac{1}{4}\sqrt{3} \quad = 23.413224^\circ\ldots \\ &\qquad\qquad\qquad = \tfrac{1}{2}\arctan \tfrac{8}{13}\sqrt{3} \quad < \alpha_{min} = \tfrac{1}{2}\arctan \tfrac{8}{12}\sqrt{3} \end{aligned}$$

12 Terdragon Graph

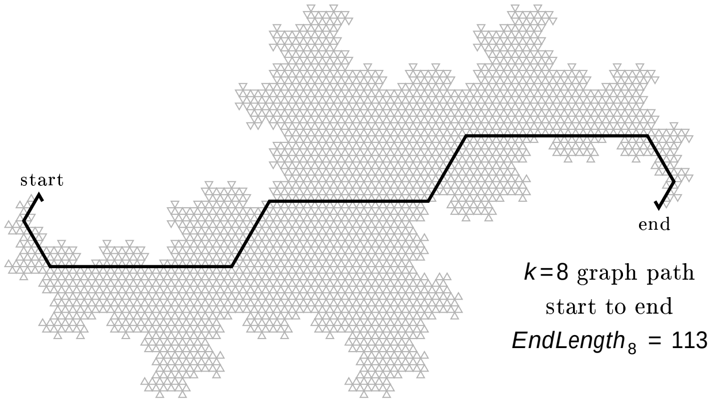
The terdragon as a graph has an Euler path from start to end (traverse all edges exactly once) simply by its construction.

There is no Hamiltonian path start to end (visit all vertices exactly once) for $k \geq 3$ since the vertices in hanging triangles cannot be visited without repeating the vertex they attach to. There is no such path in k=2 either.

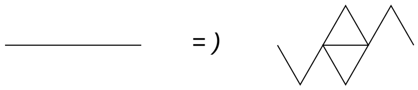
Theorem 40.

$$\mathit{EndLength}_k = \begin{cases} 3 & k=1 \\ \frac{1}{8}\ [11,19]\,3^{\frac{k}{2}} + 2k + [-3,-5,3,1] & k\neq 1 \end{cases} \tag{76}$$

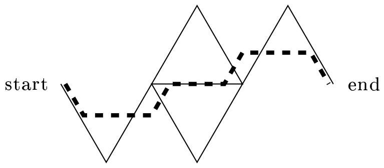
$$= 1, 3, 5, 8, 13, 22, 39, 66, 113, 194, \ldots$$



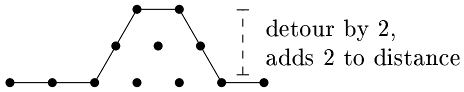
Firstly take k even and let $h = k/2$. Curve k comprises 9 sub-curves,



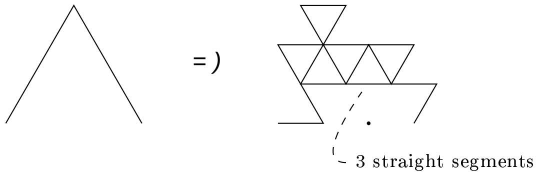
The shortest path start to end would be a straight line which is 3^h segments. But it's necessary to detour away from that midline up and down to go around the V shaped indent at start and end.



Making such a detour on a triangular grid adds a distance equal to the detour extent,



A straight line has a V indent sub-curve as shown above. Such a V comprises 18 sub-curves



The three straight lines then are V indent sub-curves again. The first and last might be partly enclosed by the angled curves adjacent to them, but the middle is not. All are located at 1 sub-curve length into the V, which is 3^{h-1}. So the sub-curves alternating straight or V down to h=0 give

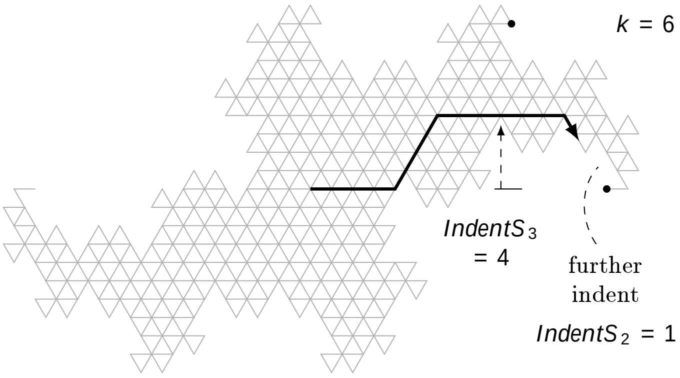
$$\mathit{IndentV}_0 = 1, \quad \mathit{IndentS}_0 = 0$$

$$\mathit{IndentV}_h = 3^{h-1} + \mathit{IndentS}_{h-1}$$

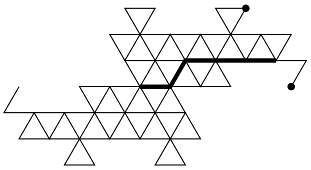
$$\begin{aligned} \mathit{IndentS}_h &= \mathit{IndentV}_{h-1} \\ &= \tfrac{1}{8}\ 3^h + [-1,5] \\ &= 0,1,1,4,10,31,91,274,820,\ldots \qquad \text{A006342} \end{aligned}$$

ternary 1010… ending 101 or 1011 for $h{-}1$ digits, $h \ge 2$

The detour around the indent reaches the centre line of the end sub-curves. They then have further perpendicular indents. This can be illustrated in the following $k{=}6$ curve. The dots are the ends of the final sub-curve. The path shown detours around $\mathit{IndentS}_3$ and reaches the centre line of that end sub-curve. The arrow shown cannot go straight but must take a further detour out.



There is always a straight path across the tops of the indent since level k comprising 81 sub-curves of $k{-}4$ is



The top horizontal lines indent at most $\mathit{IndentS}_{h-2}$ downwards and the path shown indents at most $\mathit{IndentS}_{h-2}$ up. But

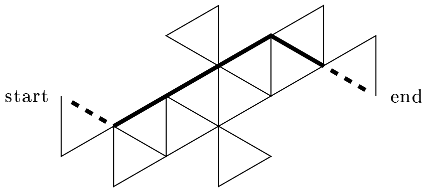
$$2\,\mathit{IndentS}_{h-2} < 3^{h-2}$$

so the top does not interfere with the path. Likewise on the diagonal up from the middle.

So for k even the distance start to end is its length 3^h plus detours at both ends which are sum of $\mathit{IndentS}$ spiralling around. This is k even of (76).

$$\mathit{EndLengthEven}_h = 3^h + 2 \sum_{j=0}^{h} \mathit{IndentS}_j$$

For k odd let $h = \lfloor k/2 \rfloor$. The shortest path start to end would be straight across stepping along the sides of rhombus shaped pairs of triangles. This is distance 2.3^h. The following diagram shows a k curve expanded 3 times to 27 sub-curves.

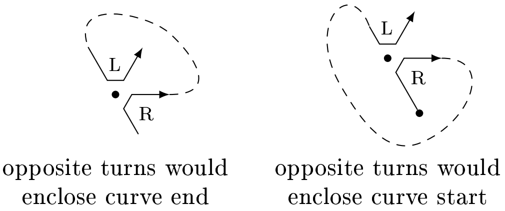


The dashed section is an indent across a V the same as for even k. A path start to end must detour around these at each end. *EndLengthEven* includes one 3^k, so adding another gives 2.3^h and two detours. This is k odd of the theorem (76).

$$\textit{EndLengthOdd}_h = 3^h + \textit{EndLengthEven}_h \qquad\qquad h \geq 1$$

This odd case effectively cuts an even path in half and inserts an extra 3^h segments which is the 3-long line in the middle of the diagram above. That middle part goes along parallel straight sides so per above the indent on its two sides do not interfere and there is a straight path of segments. ☐

When the terdragon revisits a location z, the second and third visits are the same turn as the first. This is so for any non-crossing closed curve or curve continuing infinitely and not encircling its start. An opposite turn would enclose either the end or the start,



opposite turns would enclose curve end

opposite turns would enclose curve start

When three terdragons are arranged in a triangle, the locations with right turns and the segments between them form a tree.

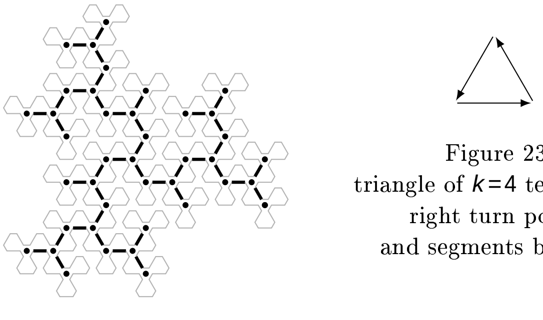
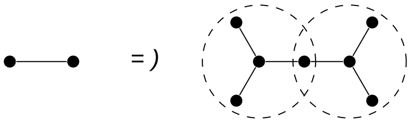


Figure 23:
triangle of $k\!=\!4$ terdragons,
right turn points
and segments between

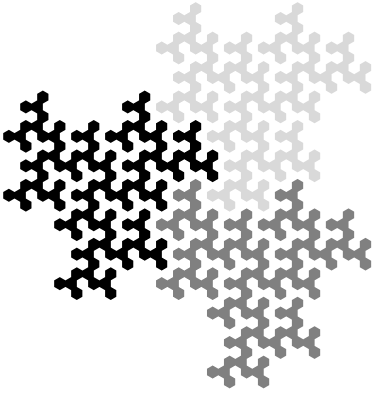
Each unit triangle has a right turn at the corner where it connects to the rest of the curve. Each unit triangle expands per figure 11 to a new right turn in the middle. The curve segments in that expansion go from the connection corners to that new right turn. An existing edge across a side becomes two segments going through the new point.

So a bottom-up expansion rule is to increase all existing vertices to degree-3 by adding new leaf vertices, and insert a new vertex in the middle of each old edge. Or equivalently a kind of star-replacement where each vertex is replaced with a claw (4-star) and each existing edge becomes a vertex in common between the new claws.



claw
replacement,
new vertex
in common

Three triangles of terdragons interlock per theorem 2 plane filling, The following diagram has each turn tree vertex drawn as a hexagon.



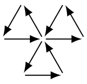


Figure 24:
3 triangles of $k{=}5$,
right turn points
(as hexagons)

The tree copy shown in black is the terdragon triangle with first segment East per figure 23. The spiralling of the terdragons directs it around to the right.

Taking only arms of the triangles at the origin continued infinitely gives the trees continuing infinitely. If curve arms are considered all going outward the 3 interlocking trees are right turns in the even arms $0, 2, 4$ and left turns in the odd arms $1, 3, 5$.

The gaps between the hexagons in figure 24 are left turn points from the terdragon triangles. They are the same tree structures as the right turns, as can be seen by rotating the pattern 60° to swap the odd and even arms and so swap which of left or right turn is taken in the arms.

The number of vertices in the tree follows from the claw replacement. Each vertex becomes 4, but in each edge there is 1 in common so

$$\begin{aligned} \mathit{TTV}_k &= 4\mathit{TTV}_{k-1} + \ \mathit{TTV}_{k-1} - 1 && \text{starting } \mathit{TTV}_1 = 1 \\ &= \tfrac{1}{2}\ 3^k - 1 && \text{A003462} \end{aligned}$$

Theorem 41.

$$k{=}1 \qquad\qquad\qquad k$$

$$\mathit{TTDegCount}(k,0) = \begin{cases} 1 & k = 1 \\ 0 & \end{cases} \qquad \text{A000007}$$

$$\mathit{TTDegCount}(k,1) = \begin{cases} 0 & k = 0,1 \\ \frac{1}{2}\ 3^{k-1} + 3 & k \geq 2 \end{cases}$$

$$= 0,0,3,6,15,42,123,366,\ldots \qquad \text{A067771}$$

$$\mathit{TTDegCount}(k,2) = \begin{cases} 0 & k = 0,1 \\ \frac{1}{2}\ 3^{k-1} - 3 & k \geq 2 \end{cases}$$

$$= 0,0,0,3,12,39,120,363,\ldots \qquad \text{A029858}$$

$$\mathit{TTDegCount}(k,3) = \begin{cases} 0 & k{=}0 \\ \frac{1}{2}\ 3^{k-1} - 1 & k \geq 1 \end{cases}$$

$$= 0,0,1,4,13,40,121,364,\ldots$$

Claw replacement gives degree-3 vertices as the preceding total vertices

$$\mathit{TTDegCount}(k,3) = \mathit{TTV}_{k-1}$$

Degree-2 vertices are likewise in each existing edge. There are $\mathit{TTV} - 1$ edges once the tree is not empty.

$$\mathit{TTDegCount}(k,2) = \mathit{TTV}_{k-1} - 1 \qquad k \geq 2$$

The claw replacement leaves only degree 1,2,3 vertices so the remainder of TTV in level k are degree-1. Or alternatively the claw replacement gives 1 degree-1 for each previous degree-2, and 2 for each previous degree-1 and 3 for each previous degree-0.

$$\begin{aligned} \mathit{TTDegCount}(k,1) &= 3\,\mathit{TTDegCount}(k{-}1,0) + 2\,\mathit{TTDegCount}(k{-}1,1) \\ &\quad + \mathit{TTDegCount}(k{-}1,2) \end{aligned} \qquad \square$$

Theorem 42. k

$$\mathit{TTdiameter}_k = \begin{cases} & k{=}0 \\ 2^k - 2 & k \geq 1 \end{cases}$$

$$= 0,2,6,14,30,62,126,\ldots \qquad k \geq 1 \qquad \text{A000918}$$

$$\mathit{TTdiameterCount}_k = \begin{cases} 0,1 & k{=}0,1 \\ 3.4^{k-2} & k \geq 2 \end{cases}$$

$$= 1,1,3,12,48,192,768,\ldots \qquad \text{A002001}$$

$$\mathit{TTdiameterEnds}_k = \begin{cases} 0,1 & k = 0,1 \\ 3.2^{k-2} & k \geq 2 \end{cases}$$

$$= 0,1,3,6,12,24,48,\ldots \qquad \text{A003945}$$

$$\mathit{TTdiameterVertices}_k = \begin{cases} 0 & k{=}0 \\ \frac{3}{4}(k{-}1)2^k + 1 & k \geq 1 \end{cases}$$

$$= 0, 1, 4, 13, 37, 97, 241, \ldots \qquad \text{A048474}$$

For any path in level $k{-}1$, the bottom-up replacement inserts 1 further edge into it for level k, so 2× the length. A path between any of those new vertices is shorter. If the path in $k{-}1$ ends at a degree-1 vertex then the replacement there has new leaf vertices attached.

A diameter must be between degree-1 vertices (otherwise could be extended). So the longest is between new leaf vertices on what was a longest path in level $k{-}1$. Starting then from diameter 0 for the single vertex of $k{=}1$,

$$\mathit{TTdiameter}_k = 2\,\mathit{TTdiameter}_{k-1} + 2 \qquad \text{starting } \mathit{TTdiameter}_1 = 0 \tag{77}$$

There are 2 new leaves at the end of the new path in k. They give endpoints, once the diameter is not 0,

$$\mathit{TTdiameterEnds}_k = 2\,\mathit{TTdiameterEnds}_{k-1} \quad \text{starting } \mathit{TTdiameterEnds}_2 = 3$$

and combinations of the 2 new at each end is 4 new paths for each existing one

$$\mathit{TTdiameterCount}_k = 4\,\mathit{TTdiameterCount}_{k-1} \qquad \text{starting } \mathit{TTdiameterCount}_2 = 3$$

For total vertices of diameters, on bottom-up replacement each existing diameter vertex has 1 new vertex towards the middle of the tree, except at the middle vertex itself. The new $\mathit{TTdiameterEnds}_k$ outer vertices are immediately adjacent to existing diameter vertices. So

$$\mathit{TTdiameterVertices}_k = 2\,\mathit{TTdiameterVertices}_{k-1} - 1 + \mathit{TTdiameterEnds}_k$$
$$\text{starting } \mathit{TTdiameterVertices}_1 = 1 \qquad \square$$

In $k = 1, 2$ all the degree-1 vertices are diameter endpoints, but in $k \geq 3$ some degree-1 are not diameter endpoints. The degree-1 vertices grow as 5^k whereas the diameter endpoints grow only as 3^k.

$$\begin{aligned} \mathit{TTdiameterEnds}_k &= \mathit{TTDegCount}(k, 1) \qquad & k &= 2, 3 \\ \mathit{TTdiameterEnds}_k &< \mathit{TTDegCount}(k, 1) \qquad & k &\geq 4 \end{aligned}$$

A top-down definition of the tree is to take the expansion of figure 11 as a level k triangle comprising 3 level $k{-}1$ triangles with a new vertex in between which is where what were left turns at connection corners are a right turn going to the next copy.

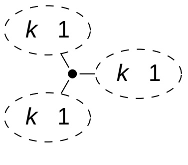


Figure 25: turns tree k as 3 copies of k 1 and new vertex in between

The connections to the $k{-}1$ are at vertices there attaining the diameter, so that the total is per (77). The three trees filling the plane can be considered like this too if the origin point is included.

The tree is half the Sierpinski triangle as a tree. That triangle has various definitions, among them is to take integer points x, y where $x \,\mathrm{BITAND}\, y = 0$. Tree edges are between points a unit distance apart.

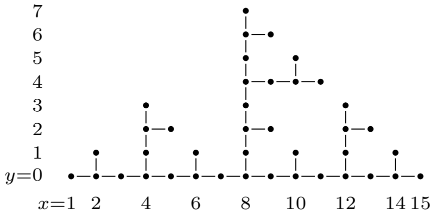


Figure 26:
Sierpinski triangle half k=4,
eighth plane $0 \quad y < x$
to depth $x+y \quad 2^k \quad 1 = 15$

This Sierpinski triangle has the same definition as figure 25. In figure 26 the middle vertex is at x=8, y=0 and the 3 sub-trees attached to it are the same.

The 3 copies in figure 25 or usual properties of the Sierpinski triangle give number of vertices $v_k = 3v_k + 1$ starting $v_0 = 0$ so $v_k = (3^k - 1)/2$.

Theorem 43.

d

d=0

$$\mathit{TTwidth}_\infty(d) = 2^{\mathit{CountOneBits}(d+1)-1} \tag{78}$$
$$= 1, 1, 2, 1, 2, 2, 4, 1, 2, 2, 4, 2, 4, 4, 8, \ldots \qquad \text{A048896}$$

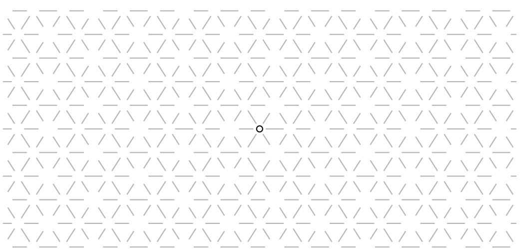
In the top-down figure 25, the new trees attach at the diameter of the first, so the first does not overlap the others.

The distance to those others is $\mathit{TTdiameter}_{k-1} + 2 = 2^{k-1}$ for $k \geq 2$, so that a depth e into them has

$$\mathit{TTwidth}(2^{k-1} + e) = 2\,\mathit{TTwidth}(e) \qquad \text{for } 0 \leq e \leq \mathit{TTdiameter}_{k-1}$$

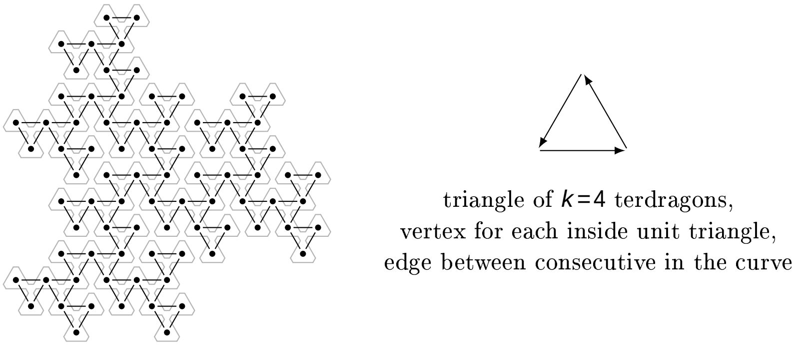
so factor 2 for a 1-bit in d. The vertex in between is at $d = 2^{k-1} - 1$ and its width is 1. That is conveniently handled by taking d+1 giving $\mathit{CountOneBits}(2^{k-1}) = 1$. Together with the initial values gives (78). ☐

$n \neq 0 \bmod 3$ are right turns when n are $n \equiv 2 \bmod 3$. They are at locations $z \equiv \omega_6 \bmod b$, from the lowest base figure expansion. This is a repeating pattern,



These locations are like k=1 trees formed from the surrounding 9 segments. Right turns with one low 0 digit on n is the same pattern with a factor of b. Those further points connect to make k=2 claws, and so on, generating the trees from a simple repeating pattern.

A related "area graph" can be formed by a vertex for each unit triangle inside the terdragon triangle and edges between those which are consecutive in the curve. Or equivalently, if corners of the curve are chamfered off to leave little gaps then edges are between unit triangles touching through those gaps.



triangle of $k{=}4$ terdragons,
vertex for each inside unit triangle,
edge between consecutive in the curve

Inside unit triangles occur in connected 3s as from the figure 11 expansion again. Each original side expands to have 2 new triangles consecutive, so the 3 new unit triangles are consecutive in pairs and so a 3-cycle in the graph.

These 3-cycles are connected like the turns tree. Each turn tree vertex is a 3-cycle and turn tree edges are where those 3-cycles share a vertex. This is a "contact triangles" form of the tree.

Or equivalently, increase all existing vertices to degree-3 by adding new leaf vertices (like the second bottom-up form above). Then the area graph is the line graph of this padded tree.

13 Fractional Locations

The location of a point $0 \le f \le 1$ along the terdragon fractal is a limit

$$\mathit{fpoint}(f) = \lim_{k\to\infty} \frac{\mathit{point}(\lfloor f.3^k \rfloor)}{b^k} \qquad \text{fractional point}$$

$n = \lfloor f.3^k \rfloor$ is the first k digits below the ternary point of f written in ternary. The location is powers b^j at each digit per (13), with rotation below each 1-digit.

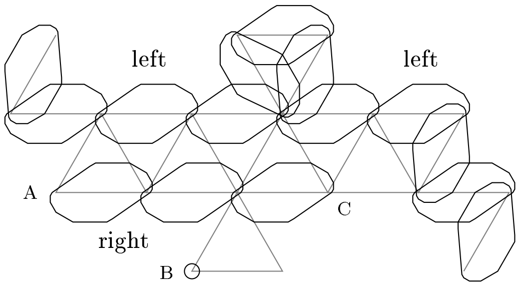
When f is rational its digits are an initial fixed part then repeating periodic part (of length at most denominator-1). The b powers are then likewise periodic and give a location as some $x{+}\omega_3 y$ with rational x, y.

If the periodic part of f has 1-digits not a multiple of 3 then there is a net rotation in the periodic part. That can be accounted for in the calculation, or repeating the part 3 times gives a multiple of 3 and so purely periodic b^j powers.

Theorem 44.

$$f = 0, 1$$

$k{=}3$ sub-curves and convex hulls around them are as follows

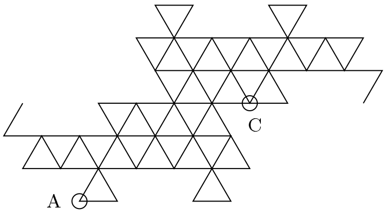


k=3 sub-curves
right and left hulls

The curve is non-crossing so all left boundary locations are within the convex hulls around the left boundary segment sub-curves.

The right boundary is within corresponding convex hulls around right boundary segments. The hulls drawn A through C, and the hanging triangle on the right side, are disjoint from the left boundary hulls. So the spiralling and curling within those parts of the right boundary never reaches the left boundary.

The sub-curves expand to k=4 as



k=4

Right boundary parts from start through A expand to the same as start through B of k=3. That leaves only sub-curves through to the corresponding new smaller A as possible both boundary. Repeating this excludes points an arbitrarily small distance away from the start, leaving only the start as both left and right boundary.

Right boundary parts end through C expand to the same as end through B of k=3. Likewise this leaves only sub-curves through to the corresponding new smaller C as possible both boundary and so anything except the end as not both left and right boundary. □

Theorem 45.

If a cut point separates start and end then it is on both left and right boundary, but from theorem 44 there are no such points.

Suppose a cut point separates a lobe from the boundary. If this point is somewhere within a sub-curve then it separates start and end of that sub-curve, but again no such point exists.

Otherwise the point is always at the start or end of some sub-curve. The only cut points in the finite iterations are the hanging triangle attachments, but they are triple-visited so by the plane filling they are not on the boundary so not cut points of the fractal. □

Theorem 46. f

f

$$\textit{fRpred}(f) = 1 \qquad 11, 12, 20$$

$$11 \qquad 0 \qquad 20 \qquad 2 \tag{79}$$

$$\textit{fLpred}(f) = \textit{fRpred}(1{-}f)$$

$$= 1 \qquad 02, 10, 11$$

$$02 \qquad 0 \qquad 11 \qquad 2$$

$$\textit{fBpred}(f) = \textit{fRpred}(f) \qquad \textit{fLpred}(f)$$

The digit pairs disallowed are the same as the finite *Rpred* and *Lpred*, but with exceptions for certain exact $f = n/3^k$. The 11 at (79) is n ending 11. The 20 is n ending 20222... which is = 21000... in the usual way. These exceptions introduce an extra state each into *Rpred* high to low figure 9.

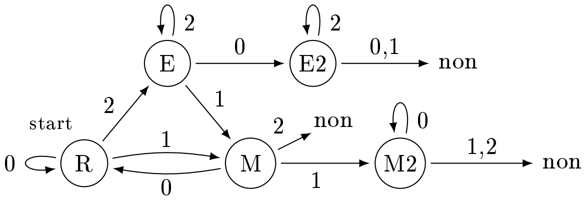
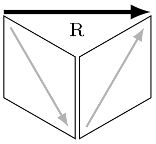


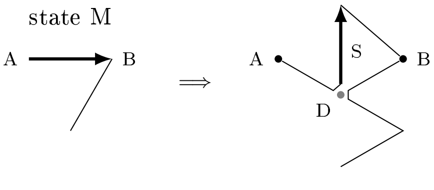
Figure 27: *fRpred*(f) by ternary digits high to low

An *Rpred* non-boundary segment has 2 enclosing segments on its right side. Since those sub-curves have no cut points, they enclose all of that side except start and end.



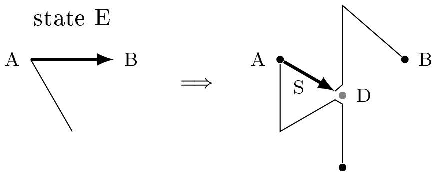
right side enclosed by 2 sub-curves when *Rpred* non-boundary

Segment start is on the right boundary when it is single or double visited and turn left (since the curve does not overlap). Single visited turn left is accepted by *Rpred* already, since there is no first segment beside it. Double-visited left turn arises from a 2-side triangle in manner of figure 11. From *Rpred* state M this is



Segment A–B expands to have S fully enclosed. This is ternary digit 1 to reach state M then digit 1 for part S. For *fRpred* the start of S is on the boundary, which is f with all 0 digits below.

A double-visited left turn from *Rpred* state E is

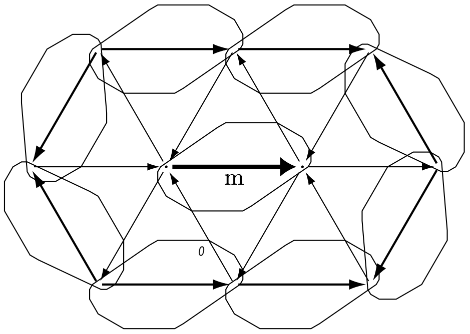


Segment A–B expands again to S fully enclosed for *Rpred*, but its end is on the boundary for *fRpred*. This is digit 2 to reach state E then digit 0 for part S. The end of S is all 2s per $.222\ldots = 1$.

A triple-visited start or end is not on the boundary, since the 6 sub-curves enclose that point per the curve plane filling.

For *fLpred* similarly with *Lpred* and double-visited turn right on the left boundary. ☐

A sub-curve m is fully surrounded by the following other sub-curves,



surrounding
convex hulls
not touching
or overlapping
hull of *m*

These surrounding sub-curves are chosen so their convex hulls do not touch the convex hull around m.

The surrounding sub-curves are continuous lines and by plane filling the fractal has no holes, so a level k sub-curve fully surrounded has any outside point a distance at least $\frac{1}{8}/\sqrt{3}^k$ away, and so all of m is non-boundary.

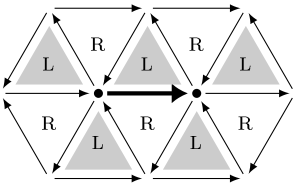
This minimum distance from m hull is measured vertically down to $P5'$ of the surrounding sub-curve. The thickness of the sub-curve there means the actual distance to the outside might be yet more, but it's only necessary to show m a non-zero distance away from all possible outside.

A given sub-curve m has some of these surrounding segments. The initial single segment $k{=}0$ has none. On expansion there are new segments around the three new sub-curves. The segments shown suffice to determine the corresponding set of segments around each new segment. A finite set of segment configurations arise and give a state machine traversed by ternary digits of f.

A fully surrounded configuration expands to fully surrounded for any next digit $0, 1, 2$. So if the digits of f ever reach fully surrounded then it remains so always. If f never reaches fully surrounded then that is an absent sub-curve at distance $\le 2/\sqrt{3}^k$ so m an arbitrarily small distance from the outside, and hence a boundary point.

$$\mathit{fBpred}(f) = \begin{cases} 0 & \text{if ever reach fully surrounded} \\ 1 & \text{if never fully surrounded} \end{cases}$$

To distinguish right and left boundary, segments of the curve always turn left or right and so divide the plane into alternating left or right side triangles (eg. as previously for area in figure 10). If a triangle has 1 or 2 segments then it is some of the outside of the curve on that side.



possible boundary triangles, left and right sides

A configuration with no R expands to no R again for next digit $0, 1, 2$. Similarly L.

$$\mathit{fRpred}(f) = \begin{cases} 0 & \text{if ever reach no } 1,2 \text{ side R triangles} \\ 1 & \text{if always a } 1,2 \text{ side R triangle} \end{cases}$$

$$\mathit{fLpred}(f) = \begin{cases} 0 & \text{if ever reach no } 1,2 \text{ side L triangles} \\ 1 & \text{if always a } 1,2 \text{ side L triangle} \end{cases}$$

Total 78 configurations arise. There are 27 with R fully enclosed and 27 with L fully enclosed. 1 configuration is both L and R fully enclosed, being the full set of segments.

Some configurations are "eventually enclosed" in the sense that some more digits from f, no matter what value, will reach enclosed. At most 3 more digits suffice for this. These configurations can be treated as enclosed since f always has further digits (low 0s if an otherwise terminating exact fraction $/3^k$). There are total 41 enclosed and eventually enclosed R, the same number for L, and 17 in common.

Some usual state machine simplification or comparison shows the result is the same as *fRpred* in figure 27. Likewise *fLpred*. ☐

This second proof does not use theorem 44 for no points on both left and right boundary. That theorem can follow mechanically from the surround state machine by getting the intersection of *fRpred* and *fLpred*. State machine manipulations show the only arbitrarily long strings matched by both are $f{=}0$ as digits .000… and $f{=}1$ as digits .222….

For computer calculation or similar, it might be decided to take only the low 0s representation of exact $f = n/3^k$. In that case state E2 is not needed in figure 27 and can go straight to non-boundary. Similarly if only low 2s representation is taken then M2 is not needed.

A given f might be known or proved to be not an exact $/3^k$ so that neither E2 nor M2 is needed, leaving just ternary without $11, 12, 20$ the same as *Rpred*.

Theorem 47. f

$$\mathit{fVisits}(f) = \begin{cases} \mathit{Visits}_k(n) & f = n/3^k \qquad n,k \\ 2 & \mathit{fNonBpred} \qquad\qquad \mathit{fBpred} \\ 1 & \end{cases}$$

An exact fraction $f = n/3^k$ is a vertex of curve k and the visits there are the same as Visits_k from (53). By plane filling those visits enclose the point so no other sub-curves touch it.

The claimed cases whole curve *fBpred* boundary or not, and sub-curve eventually or never *fBpred*, are

	whole curve	
	fBpred	*fNonBpred*
sub-curve eventually *fBpred*	1	2
sub-curve never *fBpred*	no such	1

An f which is on the boundary of some sub-curve, meaning its digits at some digit position and below are *fBpred*, might have an adjacent sub-curve like

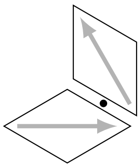


Figure 28:
f on sub-curve boundary
and adjacent other sub-curve

If it has this further sub-curve then by plane filling and no cut points the two enclose the location so visits are only those arising from the two.

If no such further sub-curve then the visits are only those arising from the f sub-curve itself. An f which is on a sub-curve boundary like this has only 1 visit because any other would be, for suitable yet smaller sub-curves, an adjacent enclosing further sub-curve like figure 28 and so not on the boundary. So in the table the first row cases are sub-curve boundary 1 or 2 visits according as whole curve boundary or not.

An f which is *fNonBpred* non-boundary, and its digits at all positions below are also *fNonBpred*, is never on the boundary of any sub-curve and so always a non-zero distance away from any other sub-curve and so just 1 visit. □

fNonBpred of a non-3^k means somewhere a ternary digit pair $02, 10, 11$ so *fNonLpred* and also somewhere $11, 12, 20$ so *fNonRpred*. Pair 11 is common to these so a 11 anywhere is *fNonBpred*.

The $\mathit{fVisits} = 2$ case is therefore at least one each $02, 10, 11$ and $11, 12, 20$, so as to be non-boundary, but only finitely many of one of them so eventually on a sub-curve boundary.

The $\mathit{fVisits} = 1$ case is the converse. Either none at all of $02, 10, 11$ or $11, 12, 20$ so whole curve *fBpred*, or infinitely many of both of them so always *fNonBpred* in all sub-curves.

The latter case, infinitely many of both, can be either rational or irrational. Suitable pairs in an infinite repeating pattern is rational, or non-repeating is irrational. The simplest rational is $f = .111\ldots = \frac{1}{2}$ which is infinite 11 pairs. This is the middle of the curve, then middle of the middle sub-curve, and so on.

It can be noted *fVisits* is not decided by initial digits of f. After some digits, a suitable exact $/3^k$ below can be $\mathit{Visits} = 3$. Or all 1s below is middle of the sub-curve $\mathit{fVisits} = 1$. Or a sub-curve boundary by suitable pairs is $\mathit{fVisits} = 2$.

Theorem 48. *fVisits*$(f)=2$
fOther(f)

fRpred disallowed pair

f	:::	122:::22	100:::00	022:::22	:::	
fOther(*f*)	:::	200:::00	022:::22	100:::00	:::	(80)
		+1	1	+1		

fRpred 11 12 20
11 1 100...
0222 1000 1222
2000 ≥ 1 *fOther*
fVisits$(f)=2$
0↔2 *fLpred*

fLpred disallowed pair

f	:::	100:::00	122:::22	200:::00	:::	
fOther(*f*)	:::	022:::22	200:::00	122:::22	:::	(81)
		1	+1	1		

All runs are maximal in the sense that they take as many of their repeating digits as possible, consistent with the next run. So 100...00 in (80) takes all 0s except one for the following 022...22.

The effect for the right side is runs begin at 1, and at 0 with non-0 below it. Or for the left side at 1, and at 2 with non-2 below it.

For the right side, the sub-curves on its right are calculated high to low as per *other* table (21). There are 2 segments on the right, but in the next expansion only one of them is used. n digits 0 or 1 use only s, and digit 2 uses only e. So a digit of *fOther* is determined by two digits of f.

R	*f* pair	00	01	02	10	11	12	20	21	22	(82)
	output	2	1	1	0	*f*2	*f*2	*f*1	0	0	

When table (21) has an "n" it is a copy of f for the output. This is shown as output f in (82) here. It occurs for the *Rpred* disallowed pairs $11, 12, 20$ so that *fOther* is unchanged above such a pair.

At the lowest disallowed pair, following the pairs there onwards in f and the output digits in (82) gives the run forms (80).

For the left side similarly, with the pairs being

L	*f* pair	00	01	02	10	11	12	20	21	22	
	output	2	2	*f*1	*f*0	*f*0	2	0	1	1	□

The left side cases are 0↔2 digit reversals of the right and its outputs. This is since the curve is the same in 180° reverse, so that $1-f$ measures back from the end and then $1-$ *fOther*$(1-f)$ measures again from the start. $1-f$ is an 0↔2 reversal.

Differences $f - \mathit{fOther}(f)$ which occur follow from the runs (80),(81). The ±1 shown under each run is the increment to add to go from f to fOther, being difference of the two ternary numbers in the runs. The runs alternate so the signs alternate.

For exact $f = n/3^k$, in theorem 5 differences are also alternating signs. For $\mathit{fVisits}(f) = 2$ by eventually sub-curve boundary there are infinite such terms, and for exact f finite terms. All differences of this form occur by choosing suitable run lengths in f.

$$\begin{aligned} f - \mathit{fOther}(f) &= \frac{1}{3^{k_0}} - \frac{1}{3^{k_1}} + \frac{1}{3^{k_2}} - \frac{1}{3^{k_3}} + \cdots \\ &\qquad \text{where } 0 > k_0 > k_1 > k_2 > \cdots \\ &= \text{fraction ternary .0 then digits 0 or 2, or 1 if all 0s below} \end{aligned}$$

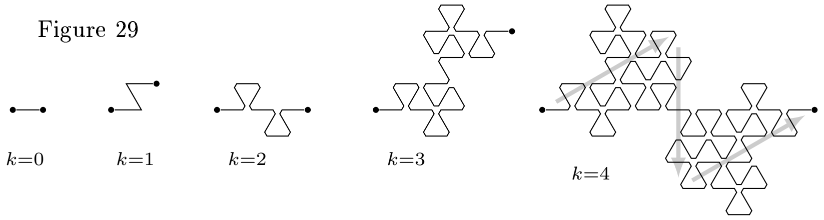
The ternary form is digit 2s for each pair $1/3^{k_0} - 1/3^{k_1}$ etc, and possible 1 like other differences from (25).

The runs have no choices within, so for the $\mathit{fVisits}(f) = 2$ case a difference determines f below its first $1/3^{k_0}$ term. When this is k_0=1 first digit, each difference occurs for just one f, fOther pair.

14 Alternate Terdragon

Davis and Knuth[3] consider also an alternate terdragon where the unfolding is to the opposite side L ↔ R in alternate expansion levels.

Figure 29



The expansion is

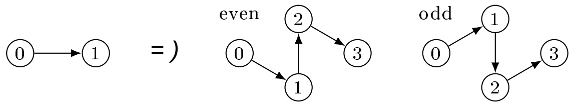


Figure 30:
alternate terdragon
expansions

The form here is taken so level k is a prefix of k+1. For unfolding this means first unfold "even" is the same as the terdragon, then second "odd" is opposite, and so on.

For segment replacement, the last replacement is to be even, so first whichever odd or even to give that last. Another equivalent is to conjugate (mirror vertically) the existing curve and replace by the terdragon "even" base each time.

Taking two expansions is a flip and flip back again so becomes a plain 9 segment replacement

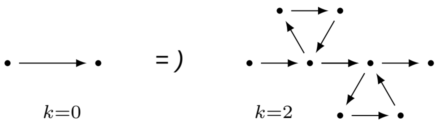


Figure 31:
alternate terdragon
two expansions

The turn sequence goes as *LowestNonZero* but the sense is flipped when that digit is at an odd position (least significant digit as position 0).

$$\mathit{AltTurn}(n) = \begin{cases} +1 & \text{if } \mathit{LowestNonZero}(n) + \mathit{CountLowZeros}(n) \equiv 1 \bmod 2 \\ -1 & \text{if } \mathit{LowestNonZero}(n) + \mathit{CountLowZeros}(n) \equiv 0 \bmod 2 \end{cases}$$

$$= -(-1)^{\mathit{LowestNonZero}(n)+\mathit{CountLowZeros}(n)}$$

$$= + - + \text{-} ++ \text{-} ++ - + \text{-} ++ - + - + \text{-} ++ - + - + \text{-} ++ \dots$$

$$\mathit{CountLowZeros}(n) = 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, \dots \quad n \geq 1 \qquad \text{A007949}$$

Or next turn, for $n \geq 0$,

$$\mathit{AltTurn}(n+1) = \begin{cases} +1 & \text{if } \mathit{LowestNonTwo}(n) + \mathit{CountLowTwos}(n) = 0 \\ -1 & \text{if } \mathit{LowestNonTwo}(n) + \mathit{CountLowTwos}(n) = 1 \end{cases}$$

$$= (-1)^{\mathit{LowestNonTwo}(n)+\mathit{CountLowTwos}(n)}$$

$$\mathit{CountLowTwos}(n) = \mathit{CountLowZeros}(n+1)$$

A turn recurrence differs from terdragon (2) in negating the $3n$ case,

$$\mathit{AltTurn}(3n) = -\mathit{AltTurn}(n), \quad \mathit{AltTurn}(3n+1) = 1, \quad \mathit{AltTurn}(3n+2) = -1 \tag{83}$$

The two expansions of figure 31 gives turns from base-9 digits of n,

$$\mathit{AltTurn}(n) = \begin{cases} +1 & \text{if } \mathit{Base9LowestNon0}(n) = 1, 4, 6, 7 \\ -1 & \text{if } \mathit{Base9LowestNon0}(n) = 2, 3, 5, 8 \end{cases}$$

$$\mathit{Base9LowestNon0}(n) = 1, 2, 3, 4, 5, 6, 7, 8, 1, 1, 2, 3, 4, \dots \quad n \geq 1 \qquad \text{A277547}$$

Predicates for left and right turns are

$$\mathit{AltTurnLpred}(n) = \begin{cases} 1 & \text{if } \mathit{AltTurn}(n) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= 1{,}0{,}0{,}1{,}0{,}1{,}1{,}0{,}1{,}1{,}0{,}0{,}1{,}0{,}1{,}1{,}0, \dots \quad n \geq 1$$

$$\mathit{AltTurnRpred}(n) = \begin{cases} 1 & \text{if } \mathit{AltTurn}(n) = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$= 0{,}1{,}1{,}0{,}1{,}0{,}0{,}1{,}0{,}0{,}1{,}1{,}0{,}1{,}0{,}0{,}1, \dots \quad n \geq 1 \qquad \text{A156595}$$

Generating functions for these in the style of (3) have L or R form powers in alternate terms,

$$\mathit{gAltTurnLpred}(x) = \sum_{k=0} \frac{x^{[1,2]\,.\,3^k}}{1 - x^{3^{k+1}}} \qquad \mathit{gAltTurnRpred}(x) = \sum_{k=0} \frac{x^{[2,1]\,.\,3^k}}{1 - x^{3^{k+1}}}$$

Combining them $\mathit{AltTurn} = \mathit{AltTurnLpred} - \mathit{AltTurnRpred}$ has both 1, 2 powers in opposite signs and alternating with k. Then like (4) cancel common factor with the denominator.

$$\mathit{gAltTurn}(x) = \sum_{k=0} \frac{(-1)^k \; x^{3^k} - x^{2.3^k}}{1 - x^{3^{k+1}}} = \sum_{k=0} \frac{(-1)^k x^{3^k}}{1 + x^{3^k} + x^{2.3^k}}$$

Direction is ±1 for each ternary digit 1 sub-part, with sign according as its digit position which is its level. Or base-9 parts $1,3,5,7$ which are directions shown in figure 31.

$$\begin{aligned}
\textit{AltDir}(n) &= \sum_{j=0}^{n-1} \textit{AltTurn}(j) \\
&= \sum_{n \text{ ternary}} \begin{cases} +1 & \text{if } \textit{digit} = 1 \text{ and even position} \\ -1 & \text{if } \textit{digit} = 1 \text{ and odd position} \\ 0 & \text{otherwise} \end{cases} \\
&= \text{count } 1,7 \;-\; \text{count } 3,5 \quad \text{of } n \text{ base-9 digits} \\
&= 0,1,0,-1,0,-1,0,1,0,1,2,1,0,1,0,1,2,1,\dots
\end{aligned}$$

The alternating unfolding is new curve end at factor b or $\overline{b}$ of the preceding,

$$\begin{aligned}
\textit{AltEnd}_k &= [b, \overline{b}].\textit{AltEnd}_{k-1} && \text{starting } \textit{AltEnd}_0 = 1 \\
&= b^{\lceil k/2 \rceil} . \overline{b}^{\lfloor k/2 \rfloor} && \text{curve end} \\
&= [1, b].3^{\lfloor k/2 \rfloor} \\
&= 1,\ 1{+}\omega_6,\ 3,\ 3{+}3\omega_6,\ 9,\ 9{+}9\omega_6,\ \dots
\end{aligned}$$

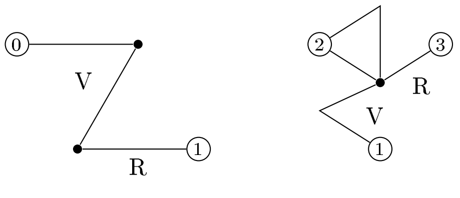
$\text{Re} = \frac{1}{2}\,\text{A038754}, \ \ \text{Im} = \frac{1}{2}\sqrt{3}\,.\,\text{A254006}$

As noted above, expansion can be treated as conjugate to flip turns then multiply b. The low digit is then the terdragon *digit* positions. So the equivalent of (14) is

$$\textit{AltPoint}(3n{+}a) = b.\overline{\textit{AltPoint}(n)} + \textit{digit}(a).\omega_3^{\textit{AltDir}(3n)}$$

This form uses $\textit{AltDir}(3n)$, which is the direction of the new first segment, to rotate $\textit{digit}(a)$ suitably. From (83) this is equivalent to $-\textit{AltDir}(n)$. That negation is a conjugate the same as $\overline{\textit{AltPoint}(n)}$.

The alternate terdragon boundary is a different shape than the terdragon, but its length is the same. That follows since the "odd" expansion in figure 30 gives R,V parts like figure 8 but opposite order, R → V,R and V → V,R



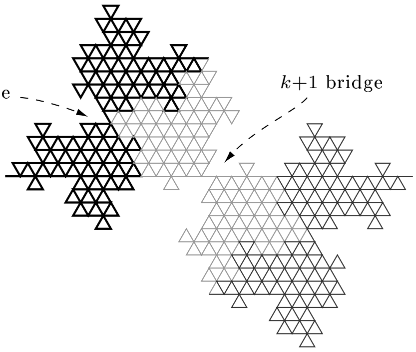
The enclosed area is likewise a different shape but the same number of unit triangles since each V expands to enclose a new unit triangle and each existing enclosed triangle expands to 3 new.

The number of single, double and triple visited points are likewise the same as terdragon S, D, T since they arise from the middles of 1,2,3 side triangles.

In figure 29 it can be seen a single bridge segment goes between two blobs of triangles. By symmetry this is the middle segment $n = \frac{1}{2}(3^k - 1)$.

k=2 is the first with a bridge segment. On expansion to k=3 there is the same form, with some additional triangles before and after, so on repeated expansion there is a single such segment in each level.

The bridge in k+1 is not the same segment number as in k. The k bridge is enclosed by the continuing curve, and the new bridge is the middle of the middle k sub-curve.

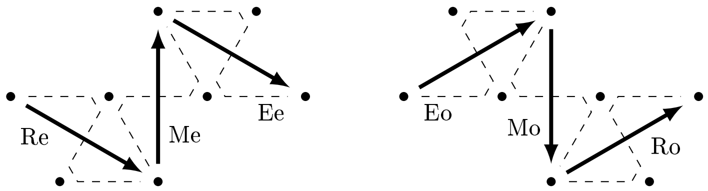


Theorem 49.
n=0

$$\mathit{AltRpred}_k(n) = \left(\begin{array}{lll} 1 & \mathit{FlipOdd}_k(n) & 10, 21, 22 \\ 0 & & \end{array}\right.$$

$$\mathit{FlipOdd}_k(n) = n \qquad k \qquad\qquad 0{\leftrightarrow}2$$

Take right boundary sub-curves in parts R,M,E similar to theorem 12.



Re,Me,Ee are even k sub-curves. They comprise odd $k-1$ sub-curves in the conjugate base pattern. Ro,Mo,Eo are odd k sub-curves. They comprise even $k-1$ sub-curves in the plain base pattern. Ro has no adjacent sub-curves, like Re has none, and then Mo and Eo are back from there. Consequently for example Re goes to sub-curves in order Eo,Mo,Ro for digit 0, 1, 2 respectively.

The expansions give the following state machine of new sub-curve type or "non" when enclosed and so not right boundary.

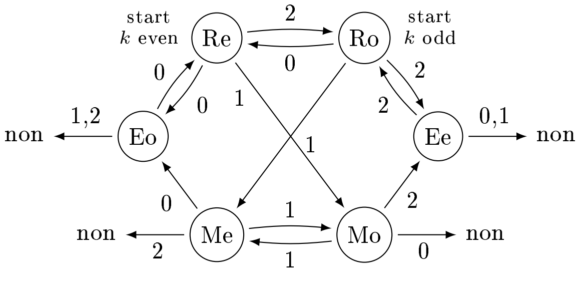


Figure 32: $\mathit{AltRpred}_k(n)$ state machine, ternary high to low

Each successive digit goes alternately to an e or o state. Transitions out of e or o are the same but digit flipped 0↔2. e is an even k sub-curve so its next digit is an odd position. Reckoning those odd positions flipped for unified R,M,E states means they are always reached by digit $0,1,2$ respectively. Transitions to "non" are then the disallowed pairs $10,21,22$. ☐

Some state machine manipulations can make a single starting state, rather than Re,Ro for k even,odd. This becomes 8 states by what is effectively simultaneous traversal with accepting or not for when no more digits (or when both reach non).

Some state machine manipulations or just following the disallowed pairs gives the following low to high form. This is a single starting state. o states are an odd number of digits below. o0 is 0 immediately below. o12 is either 1 or 2 immediately below. e states are an even number of digits below, with e01 having either 0 or 1 immediately below and e2 having 2 immediately below.

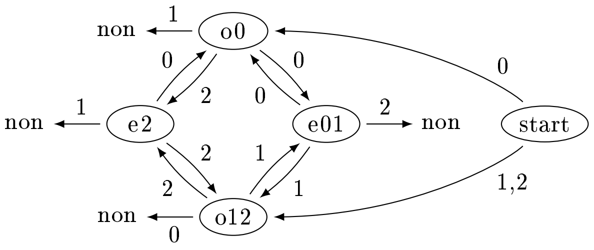
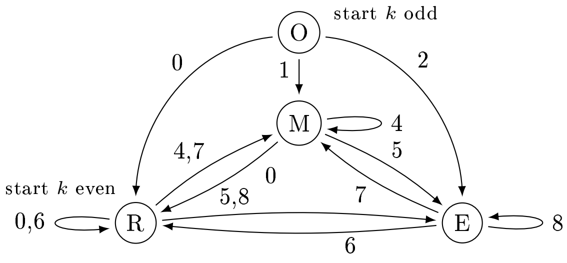


Figure 33: *AltRpred*(n) state machine, ternary low to high

A yet further approach is to take base-9 digits. Going high to low is R,M,E parts expanding per base-9 figure 31. The start state is R for even k, or an extra O for odd k and the high digit goes to R,M,E.



AltRpred$_k(n)$ state machine, base-9 high to low

The alternate terdragon is symmetric in 180° rotation, so left boundary segment numbers are the right boundary counted from the end, which means 0↔2 digit reversal. Disallowed pairs are the same by flipping at even instead of odd positions,

$$\mathit{AltLpred}_k(n) = \mathit{AltRpred}_k(3^k{-}1 - n)$$

$$= \begin{cases} 1 & \text{if } \mathit{FlipEven}_k(n) \text{ has no digit pair } 10,21,22 \\ 0 & \text{if any such pair} \end{cases}$$

$$\mathit{FlipEven}_k(n) = n \text{ in } k \text{ many ternary digits, flip } 0{\leftrightarrow}2 \text{ at even positions}$$

For the alternate terdragon continued infinitely, an infinite number of high 0 digits can be considered. After one high 0 the *AltRpred* result is unchanged.

This can be seen in low to high figure 33 where a 0 reaches "non" from o12 and otherwise 0s reach and bounce between o0 and e01.

$$\begin{aligned}\mathit{AltRpred}_\infty(n) &= \mathit{AltRpred}_k(n) \qquad \text{for } k \text{ with } 3^k > 3n \\ &= 1,0,0,0,1,1,1,1,1,0,0,0,\dots \\ =&1 \text{ at } n = 0,4,5,6,7,8,36,40,41,\dots \\ \mathit{AltLpred}_\infty(n) &= \mathit{AltLpred}_k(n) \qquad \text{for } k \text{ with } 3^k > 3n \\ &= 1,1,1,0,0,0,0,0,0,0,0,0,\dots \\ =&1 \text{ at } n = 0,1,2,12,13,17,18,19,20,\dots\end{aligned}$$

Those n which are *AltRpred* can be formed from an index m written in mixed radix ternary low and rest binary, similar to *Rn* theorem 13. Working low to high through figure 33, at each state after the start there are two digits continuing. For example at o0 either $0,2$. The binary digits choose those two.

The R,M,E states of figure 32 give a count of how many sides the triangle on the right of a segment has, like *Rsides* from (44). This is 1 or 2 for a boundary segment or 3 for a non-boundary.

$$\begin{aligned}\mathit{AltRsides}_k(n) &= \begin{cases} 1 & \text{if } \mathit{AltRpred}_k(n) \text{ state Re or Ro} \\ 2 & \text{if } \mathit{AltRpred}_k(n) \text{ state Me, Mo, Ee or Eo} \\ 3 & \text{if } \mathit{AltRpred}_k(n) \text{ state "non"} \end{cases} \\ &= 3 - \mathit{AltRpred}_k(n) . [2,1,1]_n \qquad (84) \\ \mathit{AltRsides}_\infty(n) &= \mathit{AltRsides}_k(n) \qquad \text{for } k \text{ with } 3^k > 3n \\ &= 1,3,3,3,2,2,1,2,2,3,3,3,\dots\end{aligned}$$

For (84), the least significant digit of n goes from an o to an e, and those transitions in figure 32 are 0 to Re, or $1,2$ to Me,Ee, so n mod 3 determines the reduction from 3 sides.

Since the number of 1 or 2 side triangles on the alternate terdragon boundary are the same as the terdragon boundary, $\mathit{AltRsides}_k$ is a permutation of terdragon *Rsides* and so for example the total is the same as from (46).

As noted above, the alternate terdragon number of single, double and triple visited points are the same as the plain terdragon. The argument of theorem 23 therefore gives the same Lines_k. But lines in each direction are different.

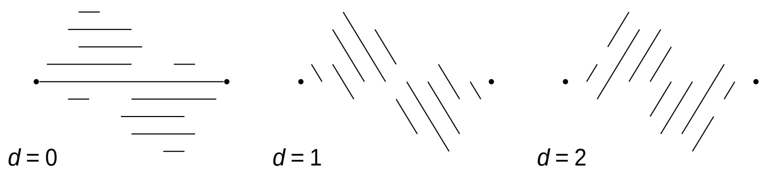


Figure 34: k=4, total Lines_4 = 31

Theorem 50. $d = 0,1,2 \;\times 120^\circ$
k

$$\mathit{AltLines}_k(0) = \tfrac{1}{3}\;\; 2^{k+1} + [1,2]$$

$$\begin{aligned}
&= 1, 2, 3, 6, 11, 22, 43, 86, 171, \dots && \text{A005578}\\
\mathit{AltLines}_k(1) &= \tfrac{1}{3}\left(2^{k+1} - [2,1]\right)\\
&= 0, 1, 2, 5, 10, 21, 42, 85, 170, \dots && \text{A000975}\\
\mathit{AltLines}_k(2) &= \tfrac{1}{3}\left(2^{k+1} - [2,4]\right)\\
&= 0, 0, 2, 4, 10, 20, 42, 84, 170, \dots && \text{A167030}
\end{aligned}$$

Single and double visited points as line endpoints of each direction can be counted the same as theorem 24 $\mathit{Lines}(d)$. The alternate terdragon expands by a conjugate then replace. The conjugate reverses directions d, so AltRTS recurrence in $-d$ and $-(d{+}1)$.

$$\mathit{AltRTS}_k(d) = \mathit{AltRTS}_{k-1}(-d) + \mathit{AltRTS}_{k-1}(-d-1)$$

Starting $\mathit{AltRTS}_0(0) = 1$ and $\mathit{AltRTS}_0(1) = \mathit{AltRTS}_0(2) = 0$ is then as follows. Case $d \equiv 2$ is the Jacobsthal numbers.

$$\mathit{AltRTS}_k(d) = \tfrac{1}{3}\left(2^k + \begin{cases} [2,1] & \text{if } d \equiv 0\\ [-1,-2] & \text{if } d \equiv 1\\ [-1,1] & \text{if } d \equiv 2 \end{cases}\right) \qquad \text{1+2 side triangles by } d$$

$$\begin{aligned}
\mathit{AltRTS}_k(0) &= 1, 1, 2, 3, 6, 11, 22, 43, 86, 171, \dots && \text{A005578}\\
\mathit{AltRTS}_k(1) &= 0, 0, 1, 2, 5, 10, 21, 42, 85, 170, \dots && \text{A000975}\\
\mathit{AltRTS}_k(2) &= 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \dots && \text{A001045}
\end{aligned}$$

Singles and doubles by direction follow from these AltRTS triangles expanding, but with relative directions from the triangles alternating with k,

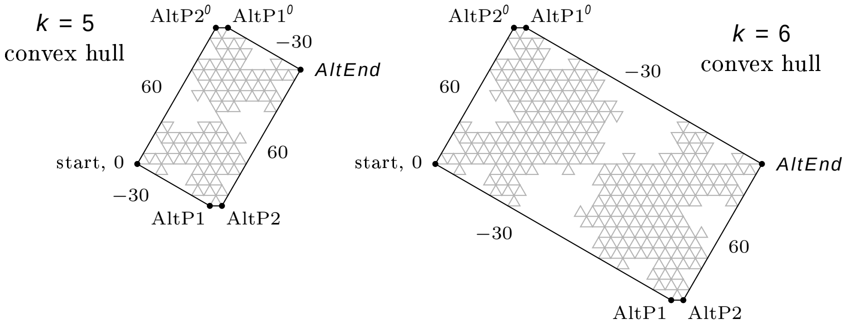
$$\begin{aligned}
\mathit{AltSD}_k(d) &= \sum_{j=0}^{k-1} \mathit{AltRTS}_j(d + (-1)^j) && \text{single, double points by } d\\
&= 2\,\mathit{AltRTS}(k,d)\ - (2 \text{ if } d{=}0)\\
\mathit{AltSD}_k(0 \text{ or } 1) &= 0, 0, 2, 4, 10, 20, 42, 84, 170, 340, \dots && \text{A167030}\\
\mathit{AltSD}_k(2) &= 0, 2, 2, 6, 10, 22, 42, 86, 170, 342, \dots && \text{A014113}
\end{aligned}$$

Then lines from single and double endpoints,

$$\mathit{AltLines}_k(d) = \tfrac{1}{2}\left(\mathit{AltSD}_k(d{+}1) + \mathit{AltSD}_k(d{+}2)\right) + (2 \text{ if } d{=}0) \qquad \square$$

$\mathit{AltLines}_k(1) = \mathit{AltLines}_k(2)$ when k even, so the same number of line, but in general the set of line lengths are not the same. In k=4 figure 34 they are the same set of lengths, but for example in k=6 they are not.

The convex hull around the alternate terdragon is a rectangle with truncated corners.



Theorem 51. $k \geq 2$

$$\mathit{AltP1}_k = \tfrac{1}{2}\overline{b}\ 3^{\lfloor k/2 \rfloor} - 1 \qquad \mathit{AltP1}'_k = \mathit{AltEnd}_k - \mathit{AltP1}_k = \mathit{AltP2}' + 1 \qquad (85)$$

$$\mathit{AltP2}_k = \mathit{AltP1}_k + 1 \qquad \mathit{AltP2}'_k = \mathit{AltEnd}_k - \mathit{AltP2}_k = \tfrac{1}{2}\omega_6\ 3^{\lceil k/2 \rceil} - 1$$

$k{=}0$ $\mathit{AltP1}_0{=}\mathit{AltP2}'_0{=}0$ $\mathit{AltP2}_0{=}$ $\mathit{AltP1}'_0{=}1$
$k{=}1$ $\mathit{AltP1}_1{=}0$ $\mathit{AltP1}'_1{=}b$

The hull around curve k is formed from the hulls around its three $k{-}1$ sub-curves. The odd and even cases are

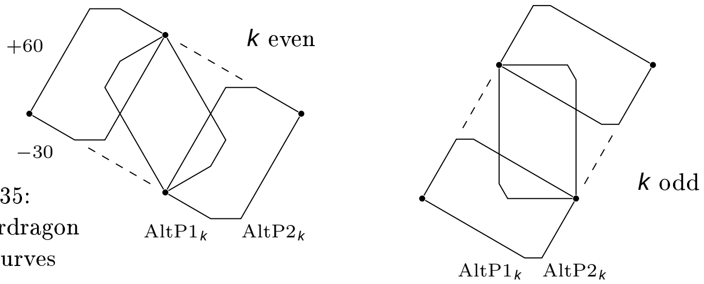


Figure 35:
alternate terdragon
hull sub-curves

AltP1, *AltP2* shift down along the $\overline{b}$ line to form k even, or are unchanged for k odd. Conversely *AltP1*$'$, *AltP2*$'$, giving (85). ☐

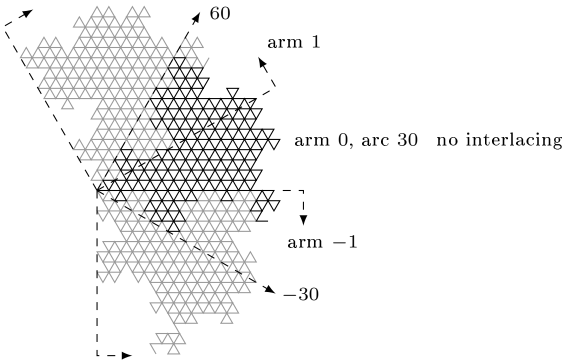
AltP1, *AltP2* are horizontal 1 segment apart. They cut off a half unit triangle from what is otherwise a rectangle. The rectangle corners are

$$\mathit{AltPC}_k = \tfrac{1}{2}\overline{b}\,3^{\lfloor k/2 \rfloor} \qquad \mathit{AltPC}'_k = \mathit{AltEnd}_k - \mathit{AltPC}_k = \tfrac{1}{2}\omega_6\,3^{\lceil k/2 \rceil}$$

The hull sub-curves in figure 35 also give the points which are on the hull boundary. The side 0 down to *AltP1* replicates in k even at a point which is $(1+\overline{\omega_3})\mathit{AltEnd}_{k-1} = \overline{b}.3^{k/2-1}$. The first of these is $k{=}2$ at $\overline{b}$. So locations $z = m.\overline{b}$ where m in ternary is digits $0, 1$ only. Similarly by replication, these are point numbers n with base-9 digits $0, 6$ only.

The side 0 up to $\mathit{AltP2}'$ replicates for k odd. Its first replication is $k{=}1$ to ω_6. So locations $z = m.\omega_6$ where m in ternary has only digits $0, 1$, and point numbers n with base-9 digits $0, 2$ only.

The hull extents are at angles −30° to +60° for total arc 90°. Adjacent arms of the curve at 60° have 30° of interlacing with their adjacent arms before and after, and 30° which is solely the arm itself.



Moment of inertia of the alternate terdragon rotating about the z axis (within the x, y plane), at its centre of mass, is the same as the terdragon I_z. Terdragon theorem 38 applies for any unfold angle, including (by symmetry) what would be negative angles at odd expansions in the alternate terdragon.

Theorem 52.

x

$$\begin{pmatrix} \mathit{AltI}_x & -\mathit{AltI}_{xy} & 0 \\ -\mathit{AltI}_{xy} & \mathit{AltI}_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \qquad \begin{aligned} \mathit{AltI}_x &= \sum y^2 \\ \mathit{AltI}_y &= \sum x^2 \end{aligned} \qquad \begin{aligned} \mathit{AltI}_{xy} &= \sum xy \\ I_z &= \sum x^2+y^2 \end{aligned}$$

I_z

$$\begin{aligned}
\mathit{AltI}_x(k) &= \tfrac{1}{156}\;\; 5.9^k \;-\; [5,6].3^{\lfloor k/2 \rfloor} \\
&= 0,\, \tfrac{1}{4},\, \tfrac{5}{2},\, \tfrac{93}{4},\, 210,\, \tfrac{7569}{4},\, \tfrac{34065}{2},\, \dots \\
\mathit{AltI}_y(k) &= \tfrac{1}{156}\;\; 8.9^k \;+\; [5,6].3^{\lfloor k/2 \rfloor} \\
&= \tfrac{1}{12},\, \tfrac{1}{2},\, \tfrac{17}{4},\, \tfrac{75}{2},\, \tfrac{1347}{4},\, \tfrac{6057}{2},\, \tfrac{109017}{4},\, \dots \\
\mathit{AltI}_{xy}(k) &= -(-1)^k.\tfrac{\sqrt{3}}{156}\;\; 2.9^k \;-\; [2,5].3^{\lfloor k/2 \rfloor} \\
&= \sqrt{3}.\left\{\; 0,\, \tfrac{1}{12},\, -1,\, \tfrac{37}{4},\, -84,\, \tfrac{3027}{4},\, -6813,\, \dots \right\}
\end{aligned}$$

Odd k comprises even $k{-}1$ sub-parts the same as figure 22 and mutual recurrences (74). Even k is odd $k{-}1$ sub-parts turning the other way. Working through those is opposite signs on AltI_{xy} for mutual recurrences

$$\mathit{AltI}_x(k) = \tfrac{3}{2}\mathit{AltI}_x(k{-}1) \;+ (-1)^k\sqrt{3}\,\mathit{AltI}_{xy}(k{-}1) \;+ \tfrac{3}{2}\mathit{AltI}_y(k{-}1) \;+ \tfrac{1}{8}9^{k-1}$$

$$\mathit{AltI}_y(k) = \tfrac{3}{2}\mathit{AltI}_x(k{-}1) \; - (-1)^k\sqrt{3}\mathit{AltI}_{xy}(k{-}1) \; + \tfrac{3}{2}\mathit{AltI}_y(k{-}1) \; + \tfrac{3}{8}9^{k-1}$$

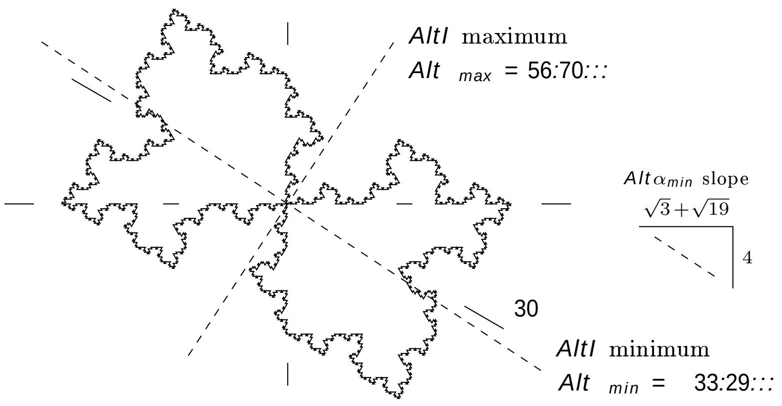
$$\mathit{AltI}_{xy}(k) = (-1)^k \quad -\tfrac{1}{2}\sqrt{3}\mathit{AltI}_x(k{-}1) \; + \tfrac{1}{2}\sqrt{3}\mathit{AltI}_y(k{-}1) \; - \tfrac{1}{8}\sqrt{3}.9^{k-1} \qquad \square$$

Principal axes are at

$$\begin{aligned} \mathit{Alt}\alpha &= \tfrac{1}{2}\arctan\frac{-2\mathit{AltI}_{xy}(k)}{\mathit{AltI}_x(k) - \mathit{AltI}_y(k)} \; + (0\,\text{or}\,\tfrac{\pi}{2}) \\ &= \tfrac{1}{2}\arctan \; (-1)^k\tfrac{4}{\sqrt{3}} - \epsilon_k \quad + (0\,\text{or}\,\tfrac{\pi}{2}) \\ \epsilon_k &= \frac{26.[-2,3].3^{\lfloor k/2\rfloor}}{3 \;\; 3.9^k + [10,12].3^{\lfloor k/2\rfloor}} \end{aligned}$$

$\epsilon_k{\to}0$ as $k{\to}\infty$ so the limit is term $\frac{1}{2}\arctan\frac{4}{\sqrt{3}}$. Factor $(-1)^k$ is conjugate negating y when k odd. Taking the fractal as repeated “unpoint”, which means the even case, limits are

$$\begin{aligned} \mathit{Alt}\alpha_{min} &\to -\tfrac{1}{2}\arctan\tfrac{4}{\sqrt{3}} \quad = -33.293387\ldots^\circ \\ \mathit{Alt}\alpha_{max} &= \mathit{Alt}\alpha_{min} + \tfrac{\pi}{2} \quad \to 56.706612\ldots^\circ \end{aligned}$$



The convex hull around the curve has limit rectangle aligned at −30°. $\mathit{Alt}\alpha_{min}$ is a little below that. Roughly speaking, there is a little more curve in the quarters clockwise from it than anti-clockwise.

For the curve scaled to unit length, unit mass, and rotated $-\mathit{Alt}\alpha_{min}$ so that x axis is the minimum, the inertia limit is

$$\begin{pmatrix} \frac{1}{24}-\frac{1}{312}\sqrt{57} & 0 & 0 \\ 0 & \frac{1}{24}+\frac{1}{312}\sqrt{57} & 0 \\ 0 & 0 & \frac{1}{12} \end{pmatrix}$$

The diameter of the alternate terdragon curve as a graph is attained between curve start and end, and for even k some additional points. In both cases all diameter paths cross the middle bridge.

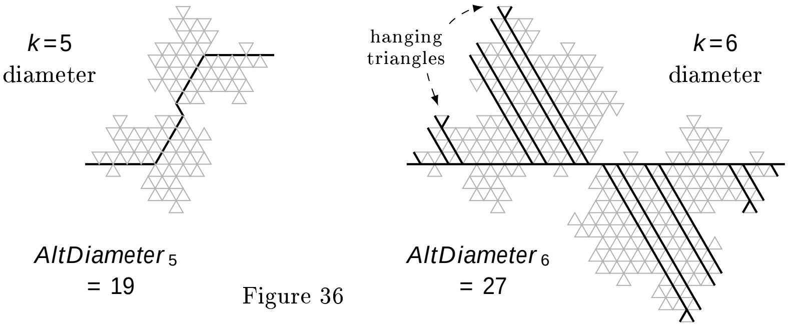


Figure 36

Theorem 53.

$$\mathit{AltDiameter}_k = \begin{cases} 3^{\lfloor k/2 \rfloor} & k \\ 2.3^{\lfloor k/2 \rfloor} + 1 & k \end{cases}$$

$$= 1, 3, 3, 7, 9, 19, 27, 55, 81, 163, \ldots$$

AltDiameter

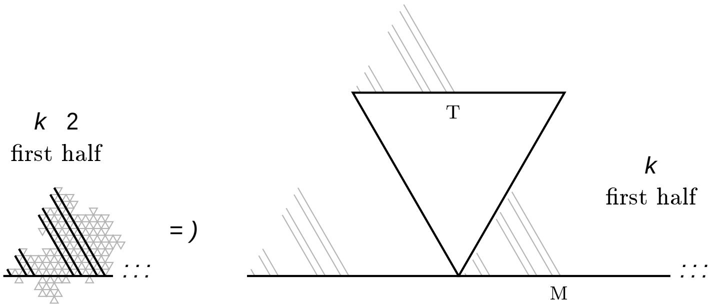
k

$$\mathit{AltDiameterEndpoints}_k = \begin{cases} 2 & k{=}0 \quad k \\ 6 & k{=}2 \\ 5.2^{k/2-1} & k \quad \geq 4 \end{cases}$$

$$= 2, 2, 6, 2, 10, 2, 20, 2, 40, \ldots$$

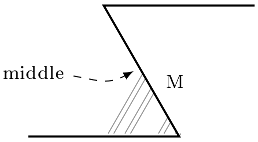
For k even, the claimed *AltDiameter* is the geometric distance through the grid. For k odd, it is geometric distance $+1$, that extra being since the middle bridge segment crossed is perpendicular to curve start to end.

To see that suitable segments of the curve exist for these distances, for k even curve start to end is a straight line and remains so on replacement by the base-9 figure 31. For points on the diagonal hull boundaries, suppose they exist in some even $k{-}2$, then the first half of curve k made from those sub-curves is



T is a $k{-}2$ sub-curve. The hull points in it have lines down to the big triangle. That triangle has straight sides and is a full grid inside by plane filling. Then sub-curve M has the same lines as T.

The longest of those lines just precedes the sub-curve midpoint, again by replication. So for k odd, the longest line in the middle sub-curve $k{-}1$ goes down to the first sub-curve.



To see no other paths attain or are shorter than the claimed *AltDiameter*, when a segment expands twice, new points are at most $+2$ from an existing,

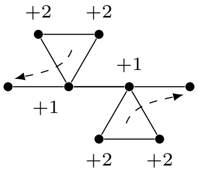


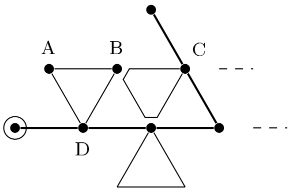
Figure 37:
new vertices distance
to expanded originals

For paths entirely within the first half of the curve, meaning up to and not including the middle bridge segment, it can be verified explicitly all are $<$ *AltDiameter* through to k=4. In k=3,4 those paths are lengths at most $3, 6$ respectively. Thereafter on each expansion points are at most $+2$ each end so an upper bound

$$\begin{aligned} h_k &= 3h_{k-2} + 4 && \text{starting } h_3 = 3,\ h_4 = 6 \\ &= [\tfrac{8}{9}, \tfrac{5}{3}].3^{\lfloor k/2 \rfloor} - 2 && \text{for } k \geq 3 \\ &< \mathit{AltDiameter}_k \end{aligned}$$

For paths crossing the middle bridge segment, points not a diameter endpoint are distance $\leq$ *AltDiameter* $- 1$ and on expansion the $+2$ at that end is new $3.{-1}{+}2 = -1$ shorter than diameter still. So only new points of figure 37 which are $\leq +2$ from what was an existing diameter endpoint need be considered.

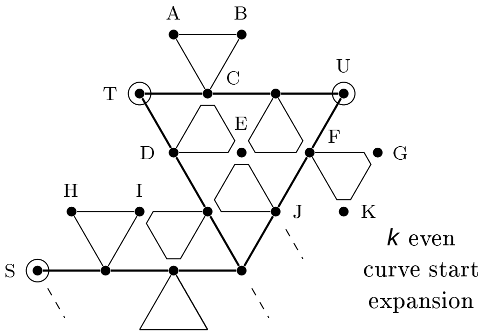
For k odd the only existing diameter endpoint is curve start and end. It expands



k odd
curve start
expansion

A,B,D are at $+1, +2$ from the start. D is on the expanded existing diameter path so is shorter. A,B can go to C. There is a horizontal line there across to the diameter path, since there is trivially in k=1 and thereafter that line passes through enclosed triangles underneath the corresponding line of $k{-}2$.

For k even,



At T, new points A,B are diameter ends per the theorem. C,E are $1, 2$ shorter down from A so are not. D is 1 shorter down from T so is not. At S, the same except its I is not a diameter end because it can go to the right to the line down from T so 1 shorter.

At U, all of U,E,F,G,K are ≤ 2 from J and that J is -3 on the diameter path of A, so U,E,F,G,K not diameter ends.

For *AltDiameterEndpoints* in even k, after k=4 the sets of 5 diagonal points and extra hanging triangle are replicated. The convex hull around the sub-curves and the replication locations mean there is no touching of those points and hanging triangles. Two sets of 5 are shown in figure 36 sample k=6.

$$\mathit{AltDiameterEndpoints}_k = 2\,\mathit{AltDiameterEndpoints}_{k-2} \qquad k \text{ even} \geq 6 \qquad \square$$

For both k even and odd there are various different paths between diameter endpoints. For k even the lines can go across as far as the last line before going down. For k odd the line shown in figure 36 can variously go up and then across.

References

[1] Jörg Arndt, “Matters Computational: Ideas, Algorithms, Source Code”, 2010, section 1.31.4 “Terdragon and Hexdragon”.
`http://www.jjj.de/fxt/fxtpage.html`

[2] Antoine-Augustin Cournot, “Solution d’un Problème d’Analyse Combinatoire”, Bulletin des Sciences Mathématiques, Physiques et Chimiques, item 34, volume 11, 1829, pages 93–97.
`http://books.google.com.au/books?id=B-v-eXuvoG4C`

[3] Chandler Davis and Donald E. Knuth, “Number Representations and Dragon Curves – I and II”, Journal of Recreational Mathematics, volume 3, number 2, April 1970, pages 66–81, and number 3, July 1970, pages 133–149.

[4] Donald E. Knuth, “Selected Papers on Fun and Games” addendum to reprint of “Number Representations and Dragon Curves”, 2010, pages 571–614.
`http://www-cs-faculty.stanford.edu/~uno/fg.html`

[5] D. E. Daykin and S. J. Tucker, “Sequences from Folding Paper”, unpublished manuscript, January 1975. Reproduced in Online Encyclopedia of

Integer Sequences (ed. N. J. A. Sloane), entry A003229.
`http://oeis.org/A003229`

[6] International Mathematical Olympiad, “Longlist”, 1992, problem 19.
`http://www.artofproblemsolving.com/Forum/resources.php?cid=18&year=1992`

[7] Benjamin Justus, “Extension of Ramus’ Identity with Applications”, Šiauliai Mathematical Seminar, 8 (16), 2013, pages 109–115.
`http://siauliaims.su.lt/index.php?option=com_content&view=article&id=338&Itemid=7`
`http://siauliaims.su.lt/pdfai/2013/Just-2013.pdf`

[8] Larry Riddle, “Classic Iterated Function Systems: Terdragon”, boundary and area, 1998–2013.
`http://ecademy.agnesscott.edu/~lriddle/ifs/heighway/terdragon.htm`
`http://ecademy.agnesscott.edu/~lriddle/ifs/heighway/terdragonBoundary.htm`
`http://ecademy.agnesscott.edu/~lriddle/ifs/heighway/terdragonArea.htm`

[9] Tiling Search database entry C07A.
`http://tilingsearch.org/HTML/data23/C07A.html`